Packing of graphs with small product of sizes

Gexin Yu
gexin.yu@vanderbilt.edu

Vanderbilt University

Joint work with A. Kostochka

An extremal problem

An extremal problem

- Question: To contain every graph with at most n vertices and at most m edges, how many edges should an n-vertex graph have?

An extremal problem

- Question: To contain every graph with at most n vertices and at most m edges, how many edges should an n-vertex graph have?
- Two graphs *pack* if one graph is a subgraph of the complement of the other one.

An extremal problem

- Question: To contain every graph with at most n vertices and at most m edges, how many edges should an n-vertex graph have?
- Two graphs *pack* if one graph is a subgraph of the complement of the other one.
- Question (in the language of packing): If G packs with a graph with m edges, how many edges are allowed in G?

Erdős-Stone Theorem

Erdős-Stone Theorem

- Erdős-Stone (1946): For each positive integer d and each $c > 0$, there is an $n_0 = n_0(d, c)$ such that for each $n \geq n_0$, every graph G of order n with at least

$$(n^2/2)(1 - 1/d) + cn^2$$

edges contains a complete $(d+1)$-partite graph H with at least vertices t in each part.

Erdős-Stone Theorem

- Erdős-Stone (1946): For each positive integer d and each $c > 0$, there is an $n_0 = n_0(d, c)$ such that for each $n \geq n_0$, every graph G of order n with at least

$$(n^2/2)(1 - 1/d) + cn^2$$

edges contains a complete $(d+1)$-partite graph H with at least vertices t in each part.

- Chvátal and Szemerédi (1981): $t \geq (\log n)/(500 \log(1/c))$

Erdős-Stone Theorem

- Erdős-Stone (1946): For each positive integer d and each $c > 0$, there is an $n_0 = n_0(d, c)$ such that for each $n \geq n_0$, every graph G of order n with at least

$$(n^2/2)(1 - 1/d) + cn^2$$

  edges contains a complete $(d+1)$-partite graph H with at least vertices t in each part.
- Chvátal and Szemerédi (1981): $t \geq (\log n)/(500 \log(1/c))$
- note that $\chi(H) \leq d + 1$. So
  if $e(G) \geq (n^2/2)(1 - 1/d) + cn^2$, then G contains all $d + 1$-colorable graphs with t vertices in each color class.

Bollobás-Eldridge's conjecture

Bollobás-Eldridge's conjecture

- Question: To contain every graph with αn ($\alpha < 1/2$) edges, how many edges can an *n*-vertex have?

Bollobás-Eldridge's conjecture

- Question: To contain every graph with αn ($\alpha < 1/2$) edges, how many edges can an n-vertex have?
- Brandt (1995) (a simplified form):
  For every $0 < \alpha < 1/2$, there exists $n_0 = n_0(\alpha)$, such that if $n > n_0$,

$$e(G_1) \leq \alpha n \quad \text{and} \quad e(G_2) \leq \frac{1}{3\sqrt{\alpha}} n^{\frac{3}{2}},$$

  then G_1 and G_2 pack.

(Bollobás-Kostochka-Nakprasit (2005), a simplified form)
Let $1/2 \leq \alpha < 1$. Let G_1 and G_2 be graphs of order $n > (\frac{40}{1-\alpha})^6$ such that

$$e(G_1) \leq \alpha n, \qquad e(G_2) \leq \frac{1}{3} n^{3/2},$$

and

$$\Delta(G_2) < n - 1 - \frac{\sqrt{n}}{\sqrt{2\alpha}(1-\alpha)}.$$

Then G_1 and G_2 pack.

Bollobás-Eldridge's packing theorem

Bollobás-Eldridge's packing theorem

- Bollobás-Eldridge (1978)
  Let G_1 and G_2 be graphs of order $n > 10$ such that $\Delta(G_1), \Delta(G_1) \leq n-2$. Then G_1 and G_2 pack

$$e(G_1) + e(G_2) \leq 2n - 3.$$

Bollobás-Eldridge's packing theorem

- Bollobás-Eldridge (1978)
  Let G_1 and G_2 be graphs of order $n > 10$ such that $\Delta(G_1), \Delta(G_1) \leq n-2$. Then G_1 and G_2 pack

$$e(G_1) + e(G_2) \leq 2n - 3.$$

- Teo and Yap (1991):
  Extend the above result to $e(G_1) + e(G_2) \leq 2n - 2$, and find all ($> 50$) non-packable pairs. Three of those are for any n.

Sauer-Spencer Theorem

Sauer-Spencer Theorem

- Sauer-Spencer (1978): Two n-vertex graphs G_1 and G_2 pack, if

$$e(G_1)e(G_2) < \binom{n}{2}.$$

Sauer-Spencer Theorem

- Sauer-Spencer (1978): Two n-vertex graphs G_1 and G_2 pack, if
$$e(G_1)e(G_2) < \binom{n}{2}.$$

- Proof.
For edge $e \in G_1$ and $f \in G_2$, let A_{ef} be the event that e is mapped to f.
Then
$$P(A_{ef}) = \frac{2(n-2)!}{n!} = \frac{1}{\binom{n}{2}}.$$
Since
$$P(\bigcup A_{ef}) \leq e(G_1)e(G_2)P(A_{ef}) < 1,$$
there is a mapping from G_1 to G_2 such that no edges are overlapped. □

Examples

Examples

- When $e(G_1)e(G_2) = \binom{n}{2}$, G_1 and G_2 don't pack if and only if

  (1) $G_1 = K_{1,n-1}$, and $G_2 = M_{n/2}$

  (2) $G_1 = K_n$ and $G_2 = K_2 \cup \overline{K_{n-2}}$

Examples

- When $e(G_1)e(G_2) = \binom{n}{2}$, G_1 and G_2 don't pack if and only if

  (1) $G_1 = K_{1,n-1}$, and $G_2 = M_{n/2}$

  (2) $G_1 = K_n$ and $G_2 = K_2 \cup \overline{K_{n-2}}$

- If allow $e(G_1)e(G_2) > \binom{n}{2}$, then there are many pair of graphs which don't pack, even the product is not much larger than $\binom{n}{2}$:

  (i) $\Delta(G_1) = n - 1$ and $\delta(G_2) \geq 1$

  (ii) $G_1 = K_3 \cup \overline{K_{n-3}}$ and G_2 has independence number two.

Main Result

For every $\epsilon > 0$, there exists N, such that for all $n > N$, if two n-vertex graphs G_1 and G_2 with

$$e(G_1)e(G_2) \leq (1-\epsilon)n^2$$

do not pack, then one of the following holds
(i) one of the graphs is K_n and the other has exactly one edge; or
(ii) $\Delta(G_1) = n-1$ and $\delta(G_2) \geq 1$; or
(iii) $G_1 = K_3 \cup \overline{K_{n-3}}$ and G_2 has independence number two.

Sketch of the proof

- Fix an $0 < \epsilon < 0.1$. Let n be large.

Sketch of the proof

- Fix an $0 < \epsilon < 0.1$. Let n be large.
- Let G_1 and G_2 satisfying $e(G_1)e(G_2) \leq (1-\epsilon)n^2$ be counterexamples.

Sketch of the proof

- Fix an $0 < \epsilon < 0.1$. Let n be large.
- Let G_1 and G_2 satisfying $e(G_1)e(G_2) \leq (1-\epsilon)n^2$ be counterexamples.
- Assume $e(G_1) \leq e(G_2)$. Then $e(G_1) < \sqrt{1-\epsilon}n < (1-\epsilon/2)n$.

Sketch of the proof

- Fix an $0 < \epsilon < 0.1$. Let n be large.
- Let G_1 and G_2 satisfying $e(G_1)e(G_2) \le (1-\epsilon)n^2$ be counterexamples.
- Assume $e(G_1) \le e(G_2)$. Then $e(G_1) < \sqrt{1-\epsilon}n < (1-\epsilon/2)n$.
- Let $\alpha = e(G_1)/n$. Then $0 < \alpha < 1 - \epsilon/2$

Sketch of the proof

- Fix an $0 < \epsilon < 0.1$. Let n be large.
- Let G_1 and G_2 satisfying $e(G_1)e(G_2) \leq (1-\epsilon)n^2$ be counterexamples.
- Assume $e(G_1) \leq e(G_2)$. Then $e(G_1) < \sqrt{1-\epsilon}n < (1-\epsilon/2)n$.
- Let $\alpha = e(G_1)/n$. Then $0 < \alpha < 1 - \epsilon/2$
- If $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$ and $\Delta_2 < n - 1 - \frac{\sqrt{n}}{1-\alpha}$, then G_1 and G_2 pack. (By Brandt's and B-K-N's Theorems)

Sketch of the proof

- Fix an $0 < \epsilon < 0.1$. Let n be large.
- Let G_1 and G_2 satisfying $e(G_1)e(G_2) \leq (1-\epsilon)n^2$ be counterexamples.
- Assume $e(G_1) \leq e(G_2)$. Then $e(G_1) < \sqrt{1-\epsilon}n < (1-\epsilon/2)n$.
- Let $\alpha = e(G_1)/n$. Then $0 < \alpha < 1-\epsilon/2$
- If $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$ and $\Delta_2 < n-1-\frac{\sqrt{n}}{1-\alpha}$, then G_1 and G_2 pack. (By Brandt's and B-K-N's Theorems)
- So we consider the following two cases:
  (1) $\Delta_2 \geq n - \frac{\sqrt{n}}{1-\alpha}$ and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$,
  (2) $e(G_2) \geq \frac{1}{3}n^{\frac{3}{2}}$.

CASE 1. $\Delta_2 \geq n - \frac{\sqrt{n}}{1-\alpha}$ and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$

- If $e(G_1) < (1 - \epsilon/2)n/2$, then since n is large and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$, by Brandt's Theorem, G_1 and G_2 pack. So we assume that $e(G_1) \geq (1-\epsilon/2)n/2$, and thus $e(G_2) \leq (2-\epsilon)n$.

CASE 1. $\Delta_2 \geq n - \frac{\sqrt{n}}{1-\alpha}$ and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$

- If $e(G_1) < (1-\epsilon/2)n/2$, then since n is large and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$, by Brandt's Theorem, G_1 and G_2 pack. So we assume that $e(G_1) \geq (1-\epsilon/2)n/2$, and thus $e(G_2) \leq (2-\epsilon)n$.
- Let $w \in V(G_2)$ with $d(w) = \Delta_2 \geq n - \frac{\sqrt{n}}{1-\alpha}$.

CASE 1. $\Delta_2 \geq n - \frac{\sqrt{n}}{1-\alpha}$ and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$

- If $e(G_1) < (1 - \epsilon/2)n/2$, then since n is large and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$, by Brandt's Theorem, G_1 and G_2 pack. So we assume that $e(G_1) \geq (1 - \epsilon/2)n/2$, and thus $e(G_2) \leq (2 - \epsilon)n$.
- Let $w \in V(G_2)$ with $d(w) = \Delta_2 \geq n - \frac{\sqrt{n}}{1-\alpha}$.
  Observation: if we can embed w with some vertex w' in G_1, then $G_2 - w$ and $G_1 - w'$ in total have at most $2n - 3$ edges, and they pack.

CASE 1. $\Delta_2 \geq n - \frac{\sqrt{n}}{1-\alpha}$ and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$

- If $e(G_1) < (1-\epsilon/2)n/2$, then since n is large and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$, by Brandt's Theorem, G_1 and G_2 pack. So we assume that $e(G_1) \geq (1-\epsilon/2)n/2$, and thus $e(G_2) \leq (2-\epsilon)n$.
- Let $w \in V(G_2)$ with $d(w) = \Delta_2 \geq n - \frac{\sqrt{n}}{1-\alpha}$.

  Observation: if we can embed w with some vertex w' in G_1, then $G_2 - w$ and $G_1 - w'$ in total have at most $2n-3$ edges, and they pack.
- If G_1 has an isolated vertex u, then $G_1 - u$ and $G_2 - w$ pack.

CASE 1. $\Delta_2 \geq n - \frac{\sqrt{n}}{1-\alpha}$ and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$

- If $e(G_1) < (1-\epsilon/2)n/2$, then since n is large and $e(G_2) < \frac{1}{3}n^{\frac{3}{2}}$, by Brandt's Theorem, G_1 and G_2 pack. So we assume that $e(G_1) \geq (1-\epsilon/2)n/2$, and thus $e(G_2) \leq (2-\epsilon)n$.
- Let $w \in V(G_2)$ with $d(w) = \Delta_2 \geq n - \frac{\sqrt{n}}{1-\alpha}$.
  Observation: if we can embed w with some vertex w' in G_1, then $G_2 - w$ and $G_1 - w'$ in total have at most $2n-3$ edges, and they pack.
- If G_1 has an isolated vertex u, then $G_1 - u$ and $G_2 - w$ pack.
- If G_1 has no isolated vertex, then G_1 must contains a tree component with $\frac{n}{\epsilon n/2} = \frac{2}{\epsilon}$ (constant!) vertices.

CASE 2. $e(G_2) \geq \frac{1}{3}n^{\frac{3}{2}}$

CASE 2. $e(G_2) \geq \frac{1}{3}n^{\frac{3}{2}}$

- We can first show that $e(G_1)$ is a constant c_0 ($c_0 < 10^5$).

CASE 2. $e(G_2) \geq \frac{1}{3}n^{\frac{3}{2}}$

- We can first show that $e(G_1)$ is a constant c_0 ($c_0 < 10^5$).
- If $c_0 = 1$, then G_1 and G_2 don't pack iff $G_2 = K_n$.

CASE 2. $e(G_2) \geq \frac{1}{3}n^{\frac{3}{2}}$

- We can first show that $e(G_1)$ is a constant c_0 ($c_0 < 10^5$).
- If $c_0 = 1$, then G_1 and G_2 don't pack iff $G_2 = K_n$.
- If $c_0 = 2$, then G_1 and G_2 don't pack only if $\delta(G_2) \geq n - 2$. Then $e(G_1)e(G_2) \geq n(n-2)$, a contradiction.

CASE 2. $e(G_2) \geq \frac{1}{3}n^{\frac{3}{2}}$

- We can first show that $e(G_1)$ is a constant c_0 ($c_0 < 10^5$).
- If $c_0 = 1$, then G_1 and G_2 don't pack iff $G_2 = K_n$.
- If $c_0 = 2$, then G_1 and G_2 don't pack only if $\delta(G_2) \geq n - 2$. Then $e(G_1)e(G_2) \geq n(n-2)$, a contradiction.
- If $c_0 = 3$ and G_1 contains a K_3, then G_1 and G_2 don't pack iff G_2 has independence number 2.

CASE 2. $e(G_2) \geq \frac{1}{3} n^{\frac{3}{2}}$

- We can first show that $e(G_1)$ is a constant c_0 ($c_0 < 10^5$).
- If $c_0 = 1$, then G_1 and G_2 don't pack iff $G_2 = K_n$.
- If $c_0 = 2$, then G_1 and G_2 don't pack only if $\delta(G_2) \geq n - 2$. Then $e(G_1)e(G_2) \geq n(n-2)$, a contradiction.
- If $c_0 = 3$ and G_1 contains a K_3, then G_1 and G_2 don't pack iff G_2 has independence number 2.
- Assume that $e(G_1) \geq 3$ and G_1 doesn't contains a K_3. The size of the complement $\overline{G_2}$ of G_2 is at least

$$\binom{n}{2} - (1-\epsilon)\frac{n^2}{c_0} \geq \frac{n^2}{2}\left(1 - \frac{1}{0.5c_0}\right) + \frac{\epsilon}{2c_0}n^2.$$

So $\overline{G_2}$ contains complete $(\lfloor 0.5c_0 \rfloor + 1)$-partite graph with $t \geq \frac{\log n}{500 \log(c_0/\epsilon)} > 10^5$ vertices in each part (Erdős-Stone).

CASE 2. $e(G_2) \geq \frac{1}{3}n^{\frac{3}{2}}$

- We can first show that $e(G_1)$ is a constant c_0 ($c_0 < 10^5$).
- If $c_0 = 1$, then G_1 and G_2 don't pack iff $G_2 = K_n$.
- If $c_0 = 2$, then G_1 and G_2 don't pack only if $\delta(G_2) \geq n - 2$. Then $e(G_1)e(G_2) \geq n(n-2)$, a contradiction.
- If $c_0 = 3$ and G_1 contains a K_3, then G_1 and G_2 don't pack iff G_2 has independence number 2.
- Assume that $e(G_1) \geq 3$ and G_1 doesn't contains a K_3. The size of the complement $\overline{G_2}$ of G_2 is at least

$$\binom{n}{2} - (1-\epsilon)\frac{n^2}{c_0} \geq \frac{n^2}{2}\left(1 - \frac{1}{0.5c_0}\right) + \frac{\epsilon}{2c_0}n^2.$$

  So $\overline{G_2}$ contains complete $(\lfloor 0.5c_0 \rfloor + 1)$-partite graph with $t \geq \frac{\log n}{500\log(c_0/\epsilon)} > 10^5$ vertices in each part (Erdős-Stone).
- So if $\chi(G_1) \leq 1 + \lfloor 0.5c_0 \rfloor$, then $\overline{G_2}$ contains G_1, that is, G_1 and G_2 pack.

To extend further......

To extend further......

- More exceptions. We don't even know the complete non-packable pairs with $e(G_1)e(G_2) < n^2$ for large n. Here are some of them:
  $G_1 = G_2 = K_3 \cup K_{1,n-4}$.
  $G_1 = K_{1,n-2} \cup K_1$ and G_2 is 2-regular.
  $G_1 = K_{1,n-3} \cup K_2$, n is divisible by 3, and $G_2 = K_3 \cup \ldots \cup K_3$.
  $G_1 = K_4 \cup \overline{K_{n-4}}$, and $G_2 = 3K_{n/3}$.

To extend further......

- More exceptions. We don't even know the complete non-packable pairs with $e(G_1)e(G_2) < n^2$ for large n. Here are some of them:
  $G_1 = G_2 = K_3 \cup K_{1,n-4}$.
  $G_1 = K_{1,n-2} \cup K_1$ and G_2 is 2-regular.
  $G_1 = K_{1,n-3} \cup K_2$, n is divisible by 3, and $G_2 = K_3 \cup \ldots \cup K_3$.
  $G_1 = K_4 \cup \overline{K_{n-4}}$, and $G_2 = 3K_{n/3}$.
- A few techniques in our proof doesn't apply when $e_1 e_2 < n^2$.

Thank you!