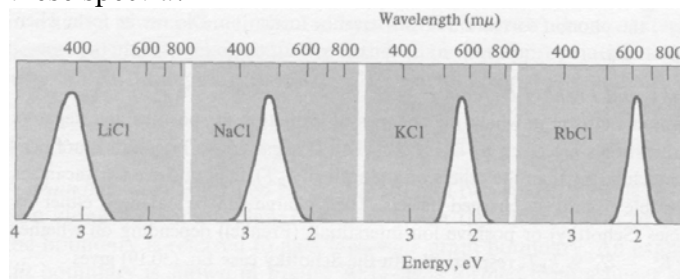


PHYSICS 225A — FALL SEMESTER 2003 — EXAMINATION 1

1. *The uncertainty principle.* **20 pts** Part 1.1 requires real numbers, Part 1.2 only the correct expressions, and some words of explanation. Two or three significant figures is plenty!
- 1.1. In the illustrations below are emission spectra from *F*-center electrons in several alkali chloride compounds; the radiation results from the transition from the first excited state to the ground state in each case. What is the range of excited-state lifetimes represented by these spectra?



The width of emission lines from the *F*-centers is a measure of the energy uncertainty in the upper electronic level; if the energy were completely determined, you would see an absolutely sharp line. The usual way of expressing this energy uncertainty is by measuring the width of the line at half its maximum amplitude. Applying this measure to the various spectra above, we find $\Delta E(\text{LiCl})=0.65$ eV, $\Delta E(\text{NaCl})=0.57$ eV, $\Delta E(\text{KCl})=0.43$ eV, $\Delta E(\text{RbCl})=0.29$ eV. Using the energy-time uncertainty relation $\Delta E \cdot \Delta t = \hbar/2\pi$, and $\hbar = 4.136 \cdot 10^{-15}$ eV·s, this gives a range of uncertainties Δt from $2.3 \cdot 10^{-15}$ s to $1.0 \cdot 10^{-15}$ s.

- 1.2. A non-relativistic microscopic particle moves freely along the *x*-axis. At some instant $t=0$, the position of the particle is measured and is found to be uncertain by Δx_0 . What is the uncertainty in the measured position of the particle Δx at some later time t ? How is this uncertainty related to the initial uncertainty Δx_0 and to the time at which the later measurement is made? What difference does this make to future experiments on this particle?

If the uncertainty in position initially is uncertain by Δx_0 , then the initial momentum is uncertain by $\Delta p_0 = \hbar/\Delta x_0$. In a time t , the particle can travel a distance $x = v_x t = (p_x/m)t$. The uncertainty in this distance is therefore $\Delta x = (\Delta p_x/m)t = (\hbar/\Delta x_0 \cdot m)t$. Note that the uncertainty is inversely proportional to the initial uncertainty Δx_0 and directly proportional to the elapsed time t . So the better one knows the position at the initial time, the greater the uncertainty in position later on.

2. *Potential steps and wells.* **30 pts** A quantum particle experiences the following one-dimensional potential near the origin:

$$\begin{aligned} V(x) &= 0, & x \leq -a/2 \text{ and } x \geq +a/2 \\ V(x) &= -U_0, & x \geq -a/2 \text{ and } x \leq +a/2 \end{aligned} \quad (1)$$

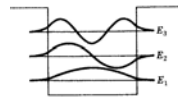
where $U_0 = 7\hbar^2\pi^2/ma^2$. Answer the following questions in a semi-quantitative way.

- 2.1. Sketch the potential diagram in one dimension. How many bound states are there? Sketch the approximate energy levels of all the bound states in the well, basing your approximation on the energy levels of the infinite square well. Where do the energy levels of this finite square well lie with respect to those of the infinite well?

Using the approximation of the infinite square well, we expect the energy levels to be close to those values, which are $E_n = (\hbar^2\pi^2/2ma^2)n^2$. The largest value of n for which the level still is below the top of the finite square well is $n=3$, so there are three energy levels in the square well. Since the well is finite, we expect all of the levels to lie slightly higher than those for the infinite square well; the difference is greatest for the $n=3$ level. [Some argued that the energy levels were lower, because the bottom of the infinite well was at 0 eV; however, this is an arbitrary value that can be shifted up or down without changing physics. The question is ambiguous because I did not ask for relative energy, and I did not figure this into credit.]

- 2.2. Sketch the wave functions and probability distributions for all of the bound states in the finite square well. What differences can you point out in comparison to the wave functions of the infinite square well?

The chief issue here is that the wave functions are almost identical to those of the infinite well, except that now it is possible for the bound particle to tunnel a little ways into the classically forbidden region outside the well. Hence the wave functions look like those in the picture just below, taken from David Park's *Introductory Quantum Mechanics*:



The probability amplitudes are just the squares of these functions, so they are everywhere positive definite; of course, the probability will also show a finite value in the tunneling region!

- 2.3. Now consider a quantum particle with energy $E > 0$ traveling to the right from $-\infty$. Describe what happens when it encounters the potential well? What is the wave function like, how is it built up, and does the energy of the incoming particle make a difference to your answer? [Hint: Think of a flute player and the sound resonance that occurs when blowing over the mouthpiece!]

To the left of the well, the incoming (traveling) wave has a wave number $k = \sqrt{2mE/\hbar^2}$; the wave to the right of the square well has the same wave number. Over the square well, the wave number increases (meaning that wavelength decreases) to $k = \sqrt{2m(E + U_0)/\hbar^2}$. Also at

the boundaries of the well, reflected and transmitted waves arise. Hence, to the left of the well, there is both a reflected and incident waves; to the right of the well, there is only a transmitted wave. However, within the well, there are forward- and backward-going waves, creating the potential for *standing waves*. For energies corresponding to bound states of the infinite well, such standing waves can even be enhanced. States like this are often called “resonances,” because they reflect an enhanced probability for finding the particle in the region of the well.

3. *Blackbody radiation and Compton scattering.* **25 pts** This problem has two separate parts. Please complete both of them. Numerical answers required!

3.1. A night-vision detector has an area of 10 cm^2 and contains 10^6 pixels. Where should its peak sensitivity in wavelength occur if it is to detect humans? Assuming the detector efficiency is 10 %, and that it needs to see 100 photons per pixel, estimate the distance at which it can detect a person against a cooler night background. Be sure to state all the assumptions that go into your calculation.

The detector has to see 100 photons per pixel to register, and has an efficiency of 10%, then there have to be 1000 photons per pixel to register a signal; with 10^6 pixels, then the detector with its area of 10 cm^2 has to see 10^9 photons in that area; this corresponds to a photon flux of $10^{12} \text{ photons/m}^2$. (We assume that the detector can see photons of all wavelengths with equal efficiency.) From the Stefan-Boltzmann law, a human body at a temperature of 37C (310K) radiates an integrated intensity (over all wavelengths) of $I = \sigma T^4 \approx 500 \text{ W/m}^2$. For an average photon energy of 0.1 eV (the peak of the blackbody spectrum at that wavelength), and assuming that the average human has a surface area $\sim 2 \text{ m}^2$, this corresponds to a net photon output of about $6 \cdot 10^{22}$ photons. Working backward from the required flux at the detector, it means that we can detect photons arriving on a sphere with an area of about $2 \cdot 10^{10} \text{ m}^2$. Such a sphere has a radius of very nearly 10^5 m , implying that one can “see” a human target at a range of around 100 km. While this may seem very far away for a hand-held night-vision detector, it should be noted that the radiation from a truck motor can be picked up by an earth-orbiting satellite, so this is not as crazy as it may seem at first glance! In practice, of course, one would have to subtract a flux of background photons, significantly reducing the range of detection.

3.2. A beam of infrared photons, wavelength $1 \mu\text{m}$, collides head on with electrons in a relativistic beam (energy 20 MeV). What is the energy of the photons scattered at an angle of 30° ? Be sure to draw a sketch of the scattering process before you start!

This process is closest to the Compton scattering example studied in the text if we assume that we are observing the collision in a frame at rest in the electron. In the rest frame of the electron, the electron will appear to have a wavenumber $k_1 = 2\gamma k_0$, $\gamma = E/m_e c^2$ foreshortened by the relativistic velocity of the electron: moreover, the relativistic motion of the electron will convert angles in the rest frame of the electron into extremely narrow angles in the laboratory rest frame. Hence, we are pretty close if we just treat the problem as one of backscattering and ignore the scattering angle. That is, all photons will pretty much be scattered into the backward direction in the laboratory reference frame. The energy of the backscattered photon is given by the relativistic Doppler shift: The frequency of the photon is upshifted into the rest frame of the electron, and Doppler shifted again when it is scattered back into the laboratory frame of reference. The new photon energy, therefore, is

$$h\nu' = h\nu \sqrt{\frac{1+\beta}{1-\beta}} \sqrt{\frac{1+\beta}{1-\beta}} = h\nu \frac{1+\beta}{1-\beta}$$

where $\beta=v/c$. (See Krane, Eq. 2.22 For the relativistic Doppler effect; this is precisely analogous to the Doppler effect for sound waves, except that in this case we are dealing with light waves or photons.) The value of β can be calculated using the fact that the relativistic *total* energy (kinetic plus rest mass) is related to β by the following equations:

$$E = K + mc^2 = \frac{mc^2}{\sqrt{1-\beta^2}} \Rightarrow 1-\beta^2 = \left(\frac{mc^2}{K+mc^2}\right)^2 \Rightarrow \beta^2 = 1 - \left(\frac{mc^2}{K+mc^2}\right)^2 = 1 - 0.0006207$$

This implies a value for $b=0.99968961$. (This is one of those places where many significant figures are needed to help prevent round-off errors in subtracting numbers very nearly equal to each other. Substituting into the expression for the Doppler shift, we obtain

$$h\nu' = h\nu \frac{1+\beta}{1-\beta} = \frac{1.99968961}{0.00031039} h\nu = h\nu(6443) = 7989 \text{ eV}$$

4. *Electron in an infinite square well.* **25 pts** An *F*-center is an electron located in an alkali halide crystal in a vacancy site where the negative ion (*e.g.*, chlorine, bromine) would otherwise be. When excited from the ground state, this captive electron is metastable, and after a short time, decays back to the ground state with the emission of a photon. The illustration accompanying Problem 1 gives the spectra of the photons emitted from the alkali chlorides; fill in those numbers in the Table below. Basing your answers on those spectra and on the “particle-in-a-box” model for the trapped electron:

- 4.1. Assuming that the decay is from the first excited state to the ground state, fill in the radii of the one-dimensional (spherical) box in which the electrons find themselves, in the appropriate places in the Table below, as predicted from the “particle-in-a-box” model.

Assuming a one-dimensional model, we can calculate the dimension of the “box” in which the electrons find themselves from Eq. 5.25 in Krane:

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \Rightarrow E_2 - E_1 = h\nu = \frac{hc}{\lambda} = \frac{\hbar^2 \pi^2 (2^2 - 1^2)}{2mL^2} \Rightarrow L^2 = \frac{3h\lambda}{8mc} = \frac{3hc\lambda}{8mc^2}$$

Carrying out the indicated arithmetic, using the fact that $hc=1240 \text{ eV}\cdot\text{nm}$ and $mc^2=0.511\cdot 10^6 \text{ eV}$, we find that $L \approx 3\cdot 10^{-2} \sqrt{\lambda}$, for λ in units of nm. Using these values, we compute the radii found in the table below.

- 4.2. How the radii computed from the infinite square-well model for the *F*-center radius compare with the Cl ion radii given in the table? What explanation can you give for any discrepancies?

The calculated radii are much smaller than the Cl⁻ radii, and indeed smaller than the alkali radii as well. This can be attributed to the Coulomb forces in the crystal: the nearest neighbors of the *F*-centers are positively charged alkali ions, but just beyond them are the negatively charged

Cl ions, “pushing” the electron into smaller orbits. (The exact calculation of those Coulomb forces is a relatively formidable problem, done in solid-state physics books.) However, the radius calculated from the simple “particle-in-a-box” model does scale roughly as $0.25 \cdot r_{\text{Cl}}$.

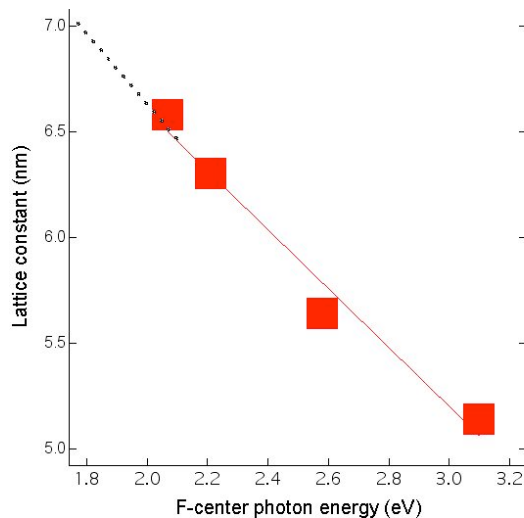
- 4.3. The lattice constant d in the NaCl structure is twice the chlorine radius in the table. Plot the lattice constant on the y -axis vs the position of the peak of the F -band radiation (from the illustration in Problem 1) on the x -axis. You may want to plot the logarithm of the two quantities to better see the trend. What does this model predict for the peak emission wavelength and lattice constant of the F -center in CsCl?

Compound	Alkali/Radius	Cl Radius	Lattice d	F -center λ	F -center Radius
LiCl	0.60 nm	2.57 nm	5.14 nm	400 nm	0.60 nm
NaCl	0.95 nm	2.82 nm	5.64 nm	480 nm	0.66 nm
KCl	1.33 nm	3.15 nm	6.30 nm	560 nm	0.71 nm
RbCl	1.48 nm	3.29 nm	6.58 nm	600 nm	0.74 nm
CsCl	1.69 nm	3.57 nm			

We assume that the infinite square model in one dimension applies. We assume that the energies of the F -center photons are given by $E_{hv} = E_2 - E_1 = (\pi^2 \hbar^2 / ma^2)(2^2 - 1^2) = (3\hbar^2 / 8ma^2)$. Using the various conversion factors for Planck’s constant and the electron mass, one finds that

$$a^2 = \frac{3\hbar^2}{8mE_{hv}} = \frac{3\hbar^2}{8(0.511 \cdot 10^6 \text{ eV}/c^2)E_{hv}} = \frac{3\hbar^2 c^2}{8(0.511 \cdot 10^6 \text{ eV})E_{hv}} = \frac{3(1240 \text{ eV} \cdot \text{nm})^2}{8(0.511 \cdot 10^6 \text{ eV})E_{hv}} = \frac{1.128}{E_{hv}} \text{ nm}^2$$

where E_{hv} is given in eV. These are the values that appear in the table. If you extend the straight line plotted through the calculated points, one finds something like the graph below.



The graph above shows the curve of F -center wavelength (or photon energy, in eV), as a function of lattice spacing, given as $2 \cdot r(\text{Cl}^-)$. If one extends the line from the four experimental points, one comes to a lattice spacing of 7.14 for CsCl, and would thus predict a wavelength of

order 1.8 eV, or 689 nm. But in fact, the F-center band for CsCl appears at 600 nm. And indeed the prediction of the lattice spacing is wrong; at CsCl, the lattice structure changes to accommodate the large Cs ion. (If you look at the handout from class, you will see that CsCl is not on the graph! That's at least one reason why ...)