Math 175 In-Class Worksheet 5

Directions: Except where indicated, merely finding the answer to a problem is not enough to receive credit. You must show how you arrived at that answer.

1) Find the value of the following integrals, switching to polar coordinates if necessary.

a) $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right)\, dy\, dx$- first try to sketch the region over which you're integrating!

b) $\displaystyle\iint_{\mathcal{R}} \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\, dx\ dy$ where $\mathcal{R}$ is the region bounded by $y = \sqrt{2x - x^2}$ and $y = -\sqrt{2x - x^2}$.

2) Time permitting, you'll finally take out $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

Two equivalent definitions of an improper integral over the whole plane $\mathbb{R}^2$ are

$$\iint_{\mathbb{R}^2} f(x,y)\ dx\ dy = \lim_{t\to\infty} \iint_{D_t} f(x,y)\ dx\ dy = \lim_{t\to\infty} \iint_{S_t} f(x,y)\ dx\ dy$$

where D_t is the disk centered at the origin with radius t and S_t is the square with vertices $(-t,t)$, $(t,-t)$, (t,t), and $(-t,-t)$, and provided these limits exist. We'll take $f(x,y)$ to be nonnegative here. For now, just assume the equivalence.

a) Using Fubini's theorem, rewrite the integral $\displaystyle\iint_{S_t} e^{-x^2-y^2}\ dx\ dy$ as a product of two one-dimensional integrals. What happens when you take the limit as $t \to \infty$?

b) Now compute the integral $\displaystyle\iint_{D_t} e^{-x^2-y^2}\ dx\ dy$ using the change of variables to polar coordinates. What happens when you take the limit as $t \to \infty$ now?

c) Find the value of the integral $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\ dx$. Then thank Lil' Wayne for polar coordinates.

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d) Once you've done part c), it's a minor step to compute $\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\ dx$. Go ahead and do this. The integrand gives what's known as a normal distribution, and is the most important kind of function in classical probability theory.

Technicalities: Remember that the improper integral $\int_{-\infty}^{\infty} f(x)\ dx$ was NOT defined as $\lim_{t\to\infty} \int_{-t}^{t} f(x)\ dx$. Rather, it was $\lim_{t\to\infty} \int_{a}^{t} f(x)\ dx + \lim_{t\to-\infty} \int_{t}^{a} f(x)\ dx$ provided BOTH limits exist and a is a random point (usually 0) that you choose.

e) Show that

$$\int_{-t}^{t} e^{-x^2}\ dx = \int_{0}^{t} e^{-x^2}\ dx + \int_{-t}^{0} e^{-x^2}\ dx = 2\int_{0}^{t} e^{-x^2}\ dx$$

so that there is no problem taking limits in the first integral.

f) Using the squeeze theorem, show that

$$\lim_{t\to\infty} \iint_{D_t} f(x,y)\ dx\ dy = \lim_{t\to\infty} \iint_{S_t} f(x,y)\ dx\ dy.$$

(Hint: we've already done one side of the inequality in some sense...)