Algebraic K-theory over the infinite dihedral group

Qayum Khan, Vanderbilt University

AMS Spring Eastern Meeting
Courant Institute of Mathematical Sciences
15 March 2008

Reference

joint work with Jim Davis and Andrew Ranicki
`http://www.arXiv.org/abs/0803.1639`

Generalized polynomial extensions of rings

Let R be a ring, and let $n \in \mathbb{Z}$. For the polynomial ring $R[x]$, Bass and Farrell determined a decomposition

$$K_n(R[x]) = K_n(R) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R; R).$$

Generalized polynomial extensions of rings

Let R be a ring, and let $n \in \mathbb{Z}$. For the polynomial ring $R[x]$, Bass and Farrell determined a decomposition

$$K_n(R[x]) = K_n(R) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R; R).$$

More generally, let $\mathscr{B}$ be an R-bimodule. For the tensor algebra $T_R\mathscr{B}$, Waldhausen determined a decomposition

$$K_n(T_R\mathscr{B}) = K_n(R) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R; \mathscr{B}).$$

Amalgamated products of rings

Let $\mathscr{B}_1, \mathscr{B}_2$ be R-bimodules. Waldhausen defined abelian groups

$$\mathrm{Nil}_*(R; \mathscr{B}_1, \mathscr{B}_2).$$

Amalgamated products of rings

Let $\mathscr{B}_1, \mathscr{B}_2$ be R-bimodules. Waldhausen defined abelian groups

$$\mathrm{Nil}_*(R; \mathscr{B}_1, \mathscr{B}_2).$$

For example, consider a pure pushout of rings

$$A = A_1 *_R A_2$$

where $A_i = R \oplus A_i'$ as R-bimodules. Then Waldhausen proved that there is a Mayer–Vietoris long exact sequence

$$\cdots \to K_n(R) \to K_n(A_1) \oplus K_n(A_2) \to \frac{K_n(A)}{\widetilde{\mathrm{Nil}}_{n-1}(R; A_1', A_2')} \xrightarrow{\partial} K_{n-1}(R) \cdots$$

THEOREM: The Nil-Nil theorem

Let R be a ring. Let $\mathscr{B}_1, \mathscr{B}_2$ be R-bimodules. Suppose the left R-module structure on $\mathscr{B}_2$ is finitely generated and projective. Then, for all $n \in \mathbb{Z}$, there is an induced isomorphism

THEOREM: The Nil-Nil theorem

Let R be a ring. Let $\mathscr{B}_1, \mathscr{B}_2$ be R-bimodules. Suppose the left R-module structure on $\mathscr{B}_2$ is finitely generated and projective. Then, for all $n \in \mathbb{Z}$, there is an induced isomorphism

$$i_* : \mathrm{Nil}_n(R; \mathscr{B}_1, \mathscr{B}_2) \longrightarrow K_n(R) \oplus \mathrm{Nil}_n(R; \mathscr{B}_1 \otimes_R \mathscr{B}_2).$$

THEOREM: The Nil-Nil theorem

Let R be a ring. Let $\mathscr{B}_1, \mathscr{B}_2$ be R-bimodules. Suppose the left R-module structure on $\mathscr{B}_2$ is finitely generated and projective. Then, for all $n \in \mathbb{Z}$, there is an induced isomorphism

$$i_* : \mathrm{Nil}_n(R; \mathscr{B}_1, \mathscr{B}_2) \longrightarrow K_n(R) \oplus \mathrm{Nil}_n(R; \mathscr{B}_1 \otimes_R \mathscr{B}_2).$$

In particular, if $\mathscr{B}_1 = \mathscr{B}_2 = R$, then there is an isomorphism

$$\widetilde{\mathrm{Nil}}_n(R; R, R) \longrightarrow \widetilde{\mathrm{Nil}}_n(R; R) \longrightarrow NK_{n+1}(R).$$

COROLLARY: Algebraic semi-splitting over the infinite dihedral group

Let $G \to D_\infty$ be an epimorphism of groups. There is an induced injective amalgamated product

$$G = G_1 *_F G_2 \quad \text{with} \quad [G_1 : F] = 2$$

and an induced HNN-extension

$$\overline{G} = F \rtimes_\alpha \mathbb{Z} \quad \text{with} \quad [G : \overline{G}] = 2.$$

COROLLARY: Algebraic semi-splitting over the infinite dihedral group

Let $G \to D_\infty$ be an epimorphism of groups. There is an induced injective amalgamated product

$$G = G_1 *_F G_2 \quad \text{with} \quad [G_1 : F] = 2$$

and an induced HNN-extension

$$\overline{G} = F \rtimes_\alpha \mathbb{Z} \quad \text{with} \quad [G : \overline{G}] = 2.$$

Then, for all rings R and $n \in \mathbb{Z}$, there is an isomorphism

$$\widetilde{\mathrm{Nil}}_n(R[F]; R[G_1 - F], R[G_2 - F]) \longrightarrow \widetilde{\mathrm{Nil}}_n(R[F]; R[F]_\alpha).$$

THEOREM: Topological semi-splitting

Consider an incompressible pushout of connected, finite CW-complexes

$$X = X_1 \cup_Y X_2 \quad \text{with} \quad \pi_1(Y) \to \pi_1(X_i) \text{ injective.}$$

Suppose $\pi_1(Y) \subset \pi_1(X_2)$ is a subgroup of finite index.

THEOREM: Topological semi-splitting

Consider an incompressible pushout of connected, finite CW-complexes

$$X = X_1 \cup_Y X_2 \quad \text{with} \quad \pi_1(Y) \to \pi_1(X_i) \text{ injective.}$$

Suppose $\pi_1(Y) \subset \pi_1(X_2)$ is a subgroup of finite index.
Then any homotopy equivalence $h' : M' \to X$ from a finite CW-complex M' is simple homotopic to a semi-split homotopy equivalence along Y.
That is, there is a finite CW-complex M, a homotopy equivalence $h : M \to X$, and a simple homotopy equivalence $f : M \to M'$ such that $h = h' \circ f$ and

$$\overline{h}_* : H_*(\overline{M_2}, N) \longrightarrow H_*(\overline{X_2}, Y)$$

is an isomorphism of $\mathbb{Z}[\pi_1(Y)]$-modules.