
Implications among linkage properties in graphs

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Joint work with Qi Liu and Doug West

Introduction

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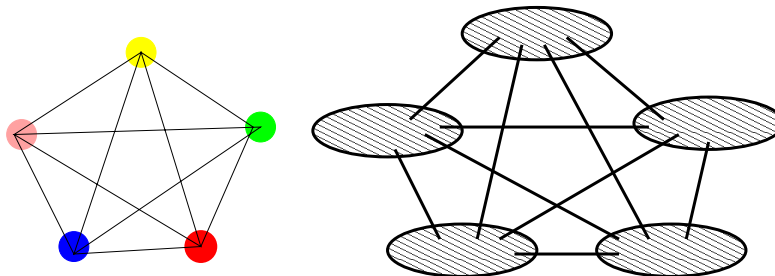
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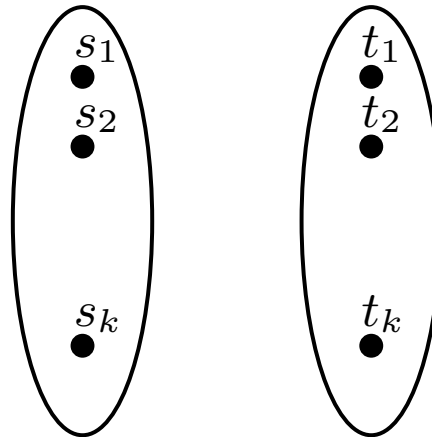
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- A possible approach to find a minor in a graph is to use what so-called linkage.

Introduction (Cont.)

- A graph is ***k*-linked** if for any $2k$ vertices $s_1, s_2, \dots, s_k, t_1, t_2, \dots, t_k$, there are k disjoint paths P_1, P_2, \dots, P_k so that P_i connects s_i to t_i .

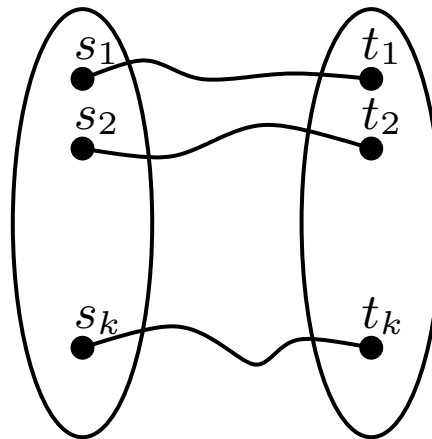
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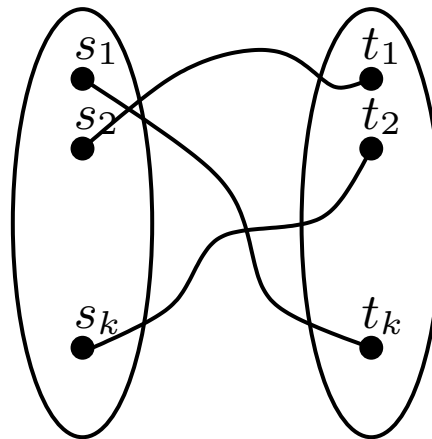
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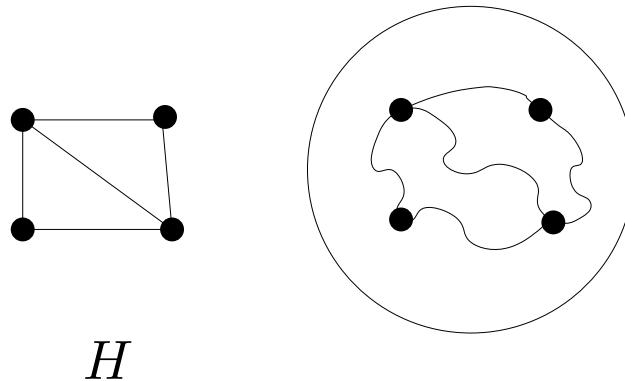
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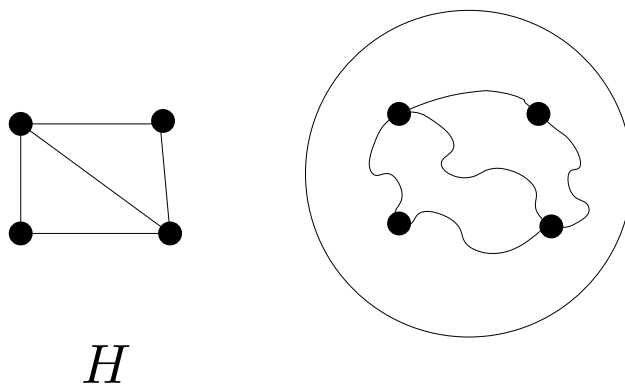
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- What we have found here is an **H-subdivision**, and in fact, we use something much stronger than having H-subdivision.

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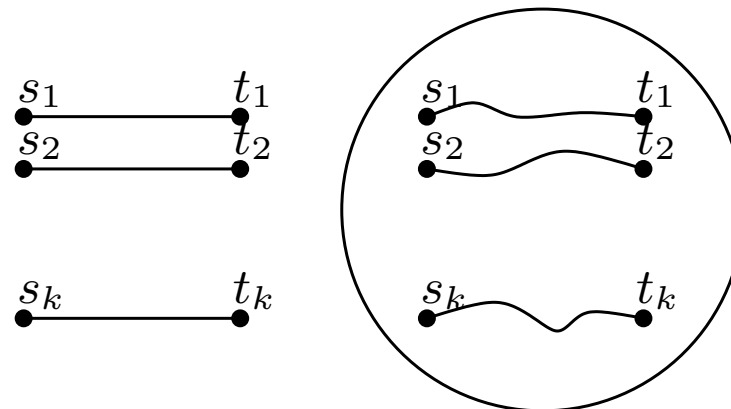
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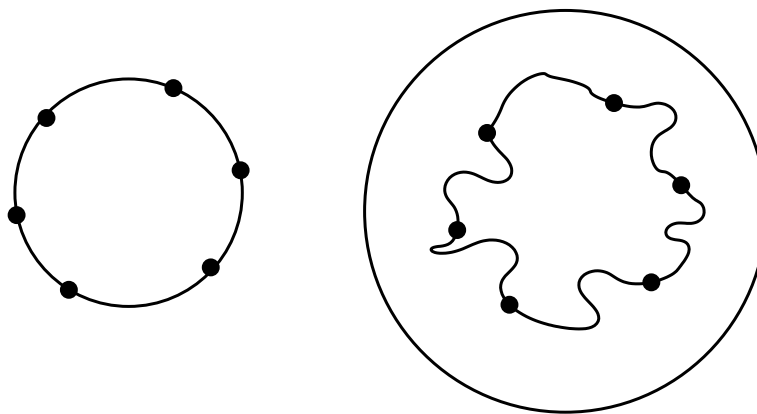


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 - A lot of well-studied notions could be stated as H -linkage for some special graph H .

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■ Sufficient degree-sum conditions (Ore-type)

- (Kostochka, Y, 2006+)

$$\sigma_2(G) \geq \begin{cases} \lceil n + (3k - 9)/2 \rceil, & \text{if } n > 2.5k - 5.5 \\ \lceil n + (3k - 8)/2 \rceil, & \text{if } 2k \leq n \leq 2.5k - 5.5 \\ 2n - 3, & \text{if } k \leq n \leq 2k - 1. \end{cases} \quad \text{suffices for}$$

k -edge graph H with min degree at least 2.

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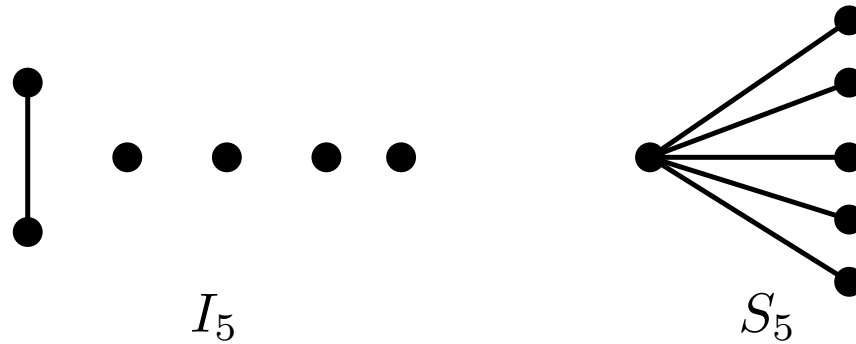
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- (X. Yu, 2003) $f(P_4) = 8$.

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- Let I_k be the graph consisting of one edge and $k - 1$ isolated vertices, and S_k be a star with k leaves.

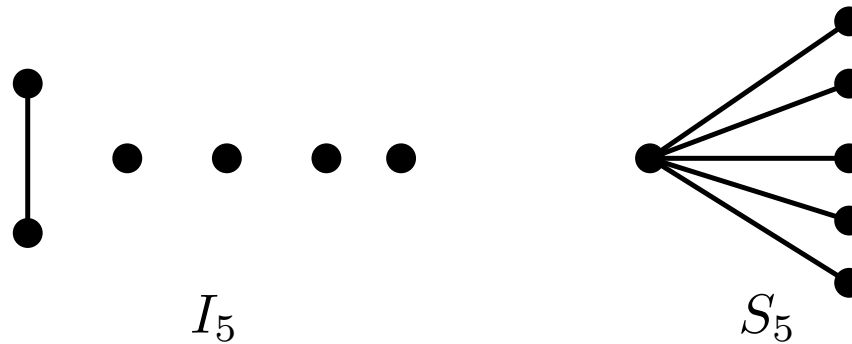
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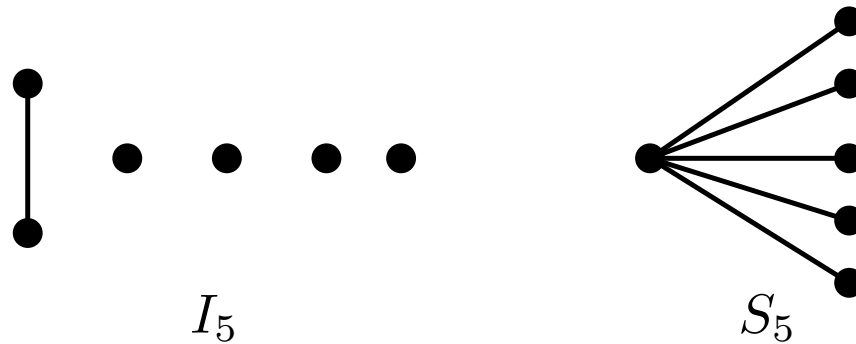
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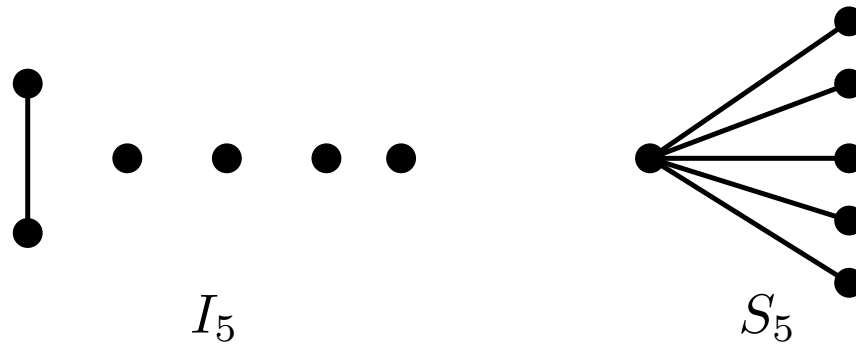
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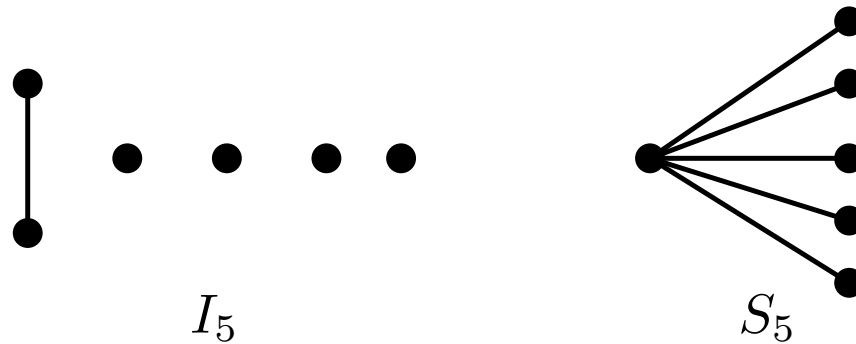
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- Any such nice properties for k -linkage, k -orderability?

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■ If H_2 has more vertices than H_1 , then $H_2 \not\leq H_1$, since $K_{n(H_2)} - e$ is not H_2 -linked.

Pairs of graphs with same order

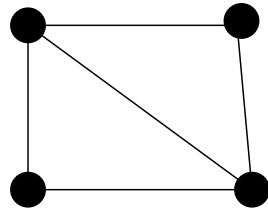
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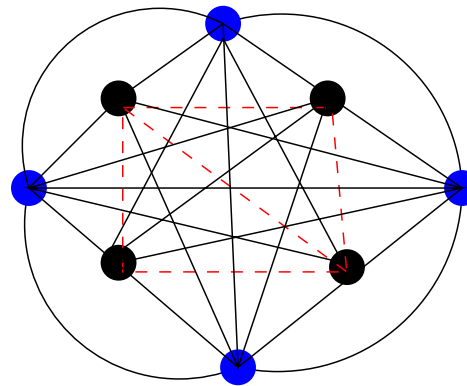
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H_2



$K_{4+5-1} - E(H_2)$

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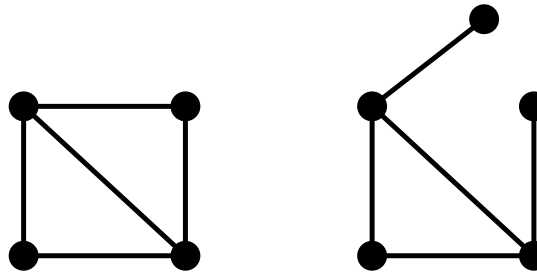
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- A corollary to above result is that if H_1 is not contained in a star, then the H_1 -linkage and H_2 -linkage properties are equivalent if and only if $H_1 \cong H_2$.

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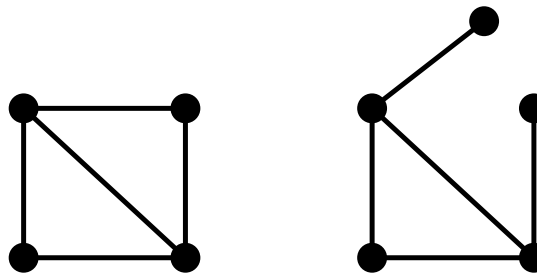
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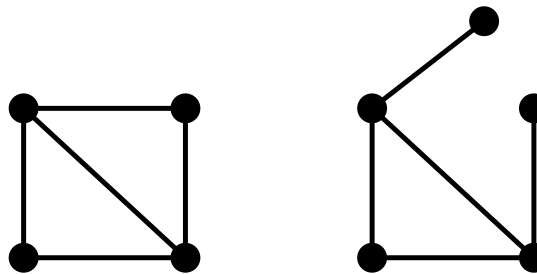
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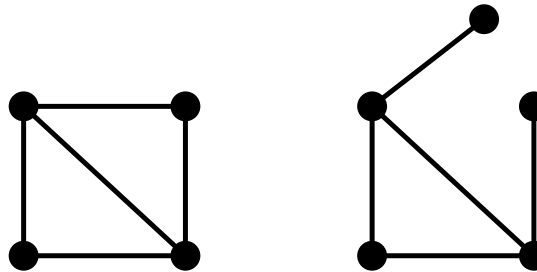
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 - k -connectivity property can be upgraded to k -linkage property by the splinter operation on S_k .

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 - $\beta(H_2) = k + 2$ also works if we require $m > 2k + 2$ or H_2 does not belong to certain family of graphs.

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 - From this result, to find implication relations, we have to restrict our attention to two graphs with a lot of common edges, or the smaller graph has small vertex cover.

Possible further research problems

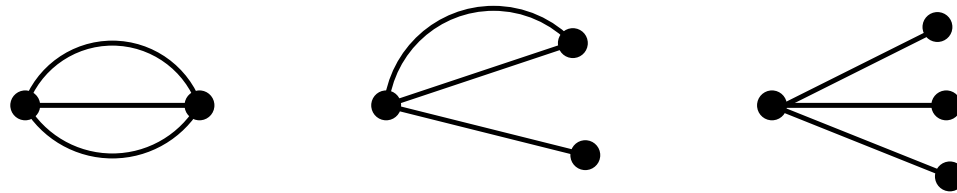
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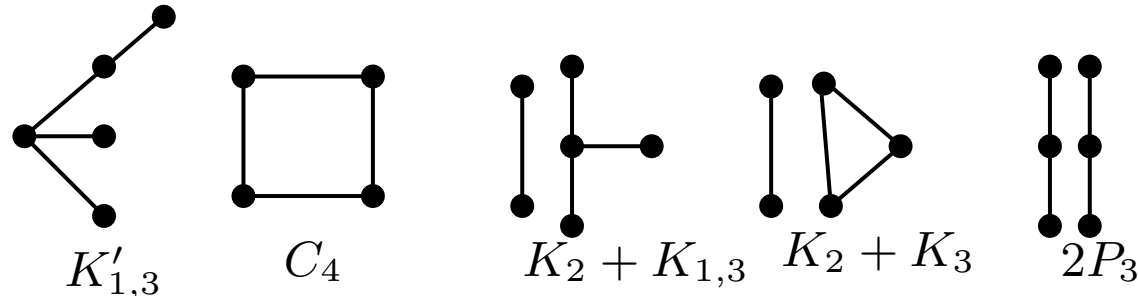


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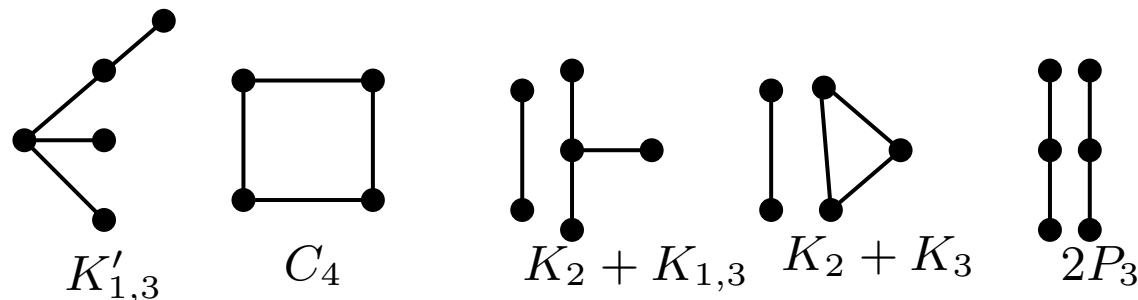
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- we know that M_k is always at the top of the poset and S_k is at the bottom, and the longest chain has length $k - 1$.

Thank you!