NOTE ON CLOSABLE DERIVATIONS ON A FINITE VON NEUMANN ALGEBRA

In this note, we work with

- a finite von Neumann algebra (M, τ);
- an ultraweakly dense $*$-subalgebra $1 \in \mathcal{D} \subset M$;
- an M-M module $\mathcal{H}$; and
- a derivation $\delta\colon M \to \mathcal{H}$ with domain $\mathcal{D}$.

A derivation is a map satisfying Leibniz's rule: $\delta(xy) = \delta(x)y + x\delta(y)$ for every $x, y \in \mathcal{D}$. We assume that the derivation δ is *closable* as an operator from L^2M into $\mathcal{H}$, with its closure written by $\bar{\delta}$. We denote by $\bar{\mathcal{D}}_{\mathrm{sa}} \subset L^2M_{\mathrm{sa}}$ the closure of $\mathcal{D}_{\mathrm{sa}}$ under the graph norm, and let

$$\bar{\mathcal{D}} = \bar{\mathcal{D}}_{\mathrm{sa}} + \sqrt{-1}\bar{\mathcal{D}}_{\mathrm{sa}} \subset \mathrm{dom}(\bar{\delta}).$$

We note that if $z \in \bar{\mathcal{D}}$, then $z^* \in \bar{\mathcal{D}}$ and there is a sequence $z_n \in \mathcal{D}$ such that $z_n \to z$ and $z_n^* \to z^*$ in the graph norm.

1. Extending the domain

Theorem (Sauvageot et al.)**.** *Let δ be a closable derivation with a self-adjoint domain. Then, $M \cap \bar{\mathcal{D}}$ is an ultraweakly dense $*$-subalgebra and $\bar{\delta}|_{M\cap\bar{\mathcal{D}}}$ is a derivation.*

Fact. Let X and Y be a locally compact Hausdorff space. For $f \in C_0(X)$ and $g \in C_0(Y)$, we define $f \otimes g \in C_0(X \times Y)$ by $(f \otimes g)(x, y) = f(x)g(y)$. By the Stone-Weierstrass theorem $\{f \otimes g\}$ spans a dense subset in $C_0(X \times Y)$. Let $*$-homomorphisms $\pi_X\colon C_0(X) \to \mathbb{B}(\mathcal{H})$ and $\pi_Y\colon C_0(Y) \to \mathbb{B}(\mathcal{H})$ be given such that their ranges commute. Then, the $*$-homomorphism $\pi_X \times \pi_Y$ defined by $(\pi_X \times \pi_Y)(f \otimes g) = \pi_X(f)\pi_Y(g)$ extends to a (continuous) $*$-homomorphism on $C_0(X \times Y)$. (This fact is known as nuclearity of $C_0(X)$, and can be proved using a partition of unity argument.)

Let Lip_0 be the space of Lipschitz functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$. Recall that f is Lipschitz if

$$\|f\|_{\mathrm{Lip}} = \sup\{\frac{|f(s) - f(t)|}{|s - t|} : s \neq t\} < \infty.$$

Lemma 1. *Let $x, y \in L^2M_{\mathrm{sa}}$ and $f \in \mathrm{Lip}_0$. Then, $f(x), f(y) \in L^2M$ and*

$$\|f(x) - f(y)\|_2 \leq \|f\|_{\mathrm{Lip}}\|x - y\|_2.$$

Proof. Let $x = \int_{\mathbb{R}} s\, dE(s)$ be the spectral decomposition of x. Then,

$$\|x\|_2^2 = \int_{\mathbb{R}} |s|^2\, d(\tau \circ E)(s) < \infty.$$

Hence $f(x) = \int_{\mathbb{R}} f(s)\, dE(s)$ and $\|f(s)| \leq \|f\|_{\mathrm{Lip}}|s|$ imply

$$\|f(x)\|_2^2 = \int_{\mathbb{R}} |f(s)|^2\, d(\tau \circ E)(s) \leq \|f\|_{\mathrm{Lip}}^2 \|x\|_2^2.$$

This proves $f(x) \in L^2M$. To prove the second assertion, we use Connes's joint distribution trick. We consider a state on $C_0(\mathbb{R} \times \mathbb{R})$ defined by

$$\sum_k f_k \otimes g_k \mapsto \tau(\sum_k f_k(x) g_k(y)) = \langle \sum_k f_k(x)\widehat{1} g_k(y), \widehat{1}\rangle_{L^2M}$$

By Riesz's representation theorem, there is a probability measure μ on $\mathbb{R} \times \mathbb{R}$ such that

$$\tau(f(x)g(y)) = \int_{\mathbb{R}\times\mathbb{R}} f(s)g(t)\, d\mu(s,t)$$

for every $f, g \in C_0(R)$. We observe that the above equation remain valid for every $f, g \in \mathrm{Lip}_0$ because $x, y \in L^2M$ implies

$$\lim_{S,T\to\infty} \int_{s>S,\, t>T} \left(s^2 + t^2\right) d\mu(s,t) = 0.$$

It follows that

$$\begin{aligned}
\|f(x) - f(y)\|_2^2 &= \tau\big(|f|^2(x) - \bar{f}(x)f(y) - f(x)\bar{f}(y) + |f|^2(y)\big) \\
&= \int_{\mathbb{R}\times\mathbb{R}} \big(|f|^2(s) - \bar{f}(s)f(t) - f(s)\bar{f}(t) + |f|^2(t)\big)\, d\mu(s,t) \\
&= \int_{\mathbb{R}\times\mathbb{R}} |f(s) - f(t)|^2\, d\mu(s,t) \\
&\leq \|f\|_{\mathrm{Lip}}^2 \int_{\mathbb{R}\times\mathbb{R}} |s-t|^2\, d\mu(s,t) \\
&= \|f\|_{\mathrm{Lip}}^2 \|x - y\|_2^2.
\end{aligned}$$

□

Let $I \subset \mathbb{R}$ be a finite interval and $f \in C^1(I)$. We define the difference quotient $\tilde{f}$ by

$$\tilde{f}(s,t) = \begin{cases} \frac{f(s)-f(t)}{s-t} & \text{if } s \neq t \\ f'(s) & \text{if } s = t \end{cases}.$$

Note that $\|\tilde{f}\|_\infty = \|f\|_{\mathrm{Lip}}$. Let $x \in M_{\mathrm{sa}}$ and $I \supset \sigma(x)$. We define a $*$-homomorphism $\pi_x \colon C(I \times I) \to \mathbb{B}(\mathcal{H})$ by

$$\pi_x(\sum_k f_k \otimes g_k)\xi = \sum_k f_k(x)\xi g_k(x).$$

Lemma 2. *Let $a \in \mathcal{D}_{sa}$ and $f \in C^1(I)_{\mathrm{sa}}$. Then, $f(x) \in \bar{\mathcal{D}}_{\mathrm{sa}} \cap M$ and*

$$\bar{\delta}(f(x)) = \pi_x(\tilde{f})\delta(x).$$

Proof. Check it for polynomials and then approximate f by polynomials. ☐

Fact. Let T be a closed operator between Hilbert spaces. Assume that $\mathrm{dom}(T) \ni x_n \to x$ and $\sup\|T(x_n)\| < \infty$. Then, there are $z_n \in \mathrm{conv}\{x_k\}_{k\geq n}$ such that $T(z_n)$ converge. In particular, $x \in \mathrm{dom}(T)$ and $\|T(x)\| \leq \limsup\|T(x_n)\|$. (Since every bounded subset is weakly pre-compact, we may assume that $T(x_n)$ converges weakly. Then, by Hahn-Banach theorem, we can find convex combinations of $T(x_n)$ which converge in norm.)

Lemma 3. *Let $x \in \bar{\mathcal{D}}_{\mathrm{sa}} \subset L^2M_{\mathrm{sa}}$ and $f \in \mathrm{Lip}_0$. Then, $f(x) \in \bar{\mathcal{D}}_{\mathrm{sa}}$ and*

$$\|\bar{\delta}(f(x))\| \leq \|f\|_{\mathrm{Lip}}\|\bar{\delta}(x)\|.$$

Proof. First assume that $x \in \mathcal{D}_{\mathrm{sa}} \subset M$. Let φ_n be a sequence of C^∞-functions on $\mathbb{R}$ such that $\varphi_n \geq 0$, $\int \varphi_n = 1$ and $\mathrm{supp}\,\varphi_n \subset [-1/n, 1/n]$. It is well-known that $f_n = f * \varphi_n$ are in C^1 and converge to f uniformly on an interval $I \supset \sigma(x)$. Moreover, $\|f_n\|_{\mathrm{Lip}} \leq \|f\|_{\mathrm{Lip}}$. Now, one has $f_n(x) \to f(x)$ in norm and

$$\|\delta(f_n(x))\| = \|\pi_x(\tilde{f}_n)\delta(x)\| \leq \|f\|_{\mathrm{Lip}}\|\delta(x)\|,$$

by Lemma 2. Hence, we are done by the above Fact. Next, let $x \in \bar{\mathcal{D}}_{\mathrm{sa}}$ and take $x_n \in \mathcal{D}_{\mathrm{sa}}$ such that $x_n \to x$ in L^2 and $\delta(x_n) \to \bar{\delta}(x)$. By Lemma 1, $f(x_n) \to f(x)$ and

$$\limsup_n \|\bar{\delta}(f(x_n))\| \leq \limsup_n \|f\|_{\mathrm{Lip}}\|\delta(x_n)\| = \|f\|_{\mathrm{Lip}}\|\bar{\delta}(x)\|.$$

By the above Fact, we are done. ☐

Lemma 4. *For every $x \in M \cap \bar{\mathcal{D}}_{\mathrm{sa}}$, there is a sequence (x_n) in $\mathcal{D}_{\mathrm{sa}}$ such that $\|x_n - x\|_2 \to 0$, $\|\delta(x_n) - \bar{\delta}(x)\| \to 0$, $\|x_n\|_M \leq \|x\|_M$; and in particular $x_n \to x$ ultraweakly.*

Proof. Let $f(t) = t \vee (-\|x\|) \wedge \|x\|$. Choose (y_n) in $\mathcal{D}_{\mathrm{sa}}$ such that $\|y_n - x\|_2 \to 0$ and $\|\delta(y_n) - \bar{\delta}(x)\| \to 0$. Then, $f(y_n) \to f(x) = x$ by Lemma 1 and

$$\limsup_n \|\bar{\delta}(f(y_n))\| \leq \limsup_n \|f\|_{\mathrm{Lip}}\|\delta(y_n)\| = \|f\|_{\mathrm{Lip}}\|\bar{\delta}(x)\| < \infty.$$

By the above Fact, we can find $x_n \in \mathrm{conv}\{f(y_k)\}_{k\geq n}$ such that $\delta(x_n)$ converges. ☐

Lemma 5. *For every $z \in \bar{\mathcal{D}}$, one has $|z| \in \bar{\mathcal{D}}$ and*

$$\|\delta(|z|)\|^2 + \|\delta(|z^*|)\|^2 \leq \|\delta(z)\|^2 + \|\delta(z^*)\|^2.$$

Proof. Check that

$$\delta^{(2)} \colon \mathbb{M}_2(M) \supset \mathbb{M}_2(\mathcal{D}) \to \mathbb{M}_2(\mathcal{H}) \cong \mathcal{H}^{\oplus 4}$$

is a closable derivation such that $\overline{\delta^{(2)}} = \bar{\delta}^{(2)}$ and $(\bar{\mathcal{D}}^{(2)})_{\mathrm{sa}} = (\bar{\mathcal{D}}_{\mathrm{sa}})^{(2)}$. Suppose $z \in \bar{\mathcal{D}}$. Then, $\tilde{z} = \left(\begin{smallmatrix} 0 & z^* \\ z & 0 \end{smallmatrix}\right) \in \bar{\mathcal{D}}^{(2)}_{\mathrm{sa}}$ and, by Lemma 3, $|\tilde{z}| = \begin{pmatrix} |z| & 0 \\ 0 & |z^*| \end{pmatrix} \in \bar{\mathcal{D}}^{(2)}_{\mathrm{sa}}$. This implies that $|z| \in \bar{\mathcal{D}}$ and $\|\delta^{(2)}(|\tilde{z}|)\| \leq \|\delta^{(2)}(\tilde{z})\|$. □

Proof of Theorem. By Lemma 5, we observe that $z \in M \cap \bar{\mathcal{D}}$ implies $|z|^2 \in M \cap \bar{\mathcal{D}}$, since $t \mapsto t^2 \wedge \|z\|^2$ is in Lip_0. Hence, by polarization identity, one has

$$x^* y = \frac{1}{4} \sum_{k=0}^{3} \sqrt{-1}^k |x + \sqrt{-1}^k y|^2 \in M \cap \bar{\mathcal{D}}$$

for every $x, y \in M \cap \bar{\mathcal{D}}$. This proves that $M \cap \bar{\mathcal{D}}$ is a $*$-subalgebra.

We are left to show $\bar{\delta}(xy) = \bar{\delta}(x)y + x\bar{\delta}(y)$ for $x, y \in M \cap \bar{\mathcal{D}}$. This can be done by two steps using Lemma 4. □

2. Quantum Markov Semigroup

In this section , we assume that the derivation δ is *real*: there is a conjugate-linear isometric involution J on $\mathcal{H}$ such that $J(x\delta(y)z) = z^*\delta(y^*)x^*$ for every $x, y, z \in \mathcal{D}$. It is easy to see that $\bar{\mathcal{D}} = \mathrm{dom}(\bar{\delta})$. Let us consider the following objects:

- $\Delta = \delta^*\bar{\delta}$; a positive self-adjoint operator on L^2M
  such that $\Delta(1) = 0$ and $\Delta J = J\Delta$, where $Jx = x^*$ for $x \in L^2M$.
- $\phi_t = e^{-t\Delta}$; a semigroup of positive contractions on L^2M
  such that $\phi_t(1) = 1$, $\phi_t J = J\phi_t$ and $\phi_t \to 1$ as $t \to 0$.
- $\rho_\alpha = \alpha(\alpha + \Delta)^{-1}$; normalized resolvent, positive contractions.

One recovers Δ from ϕ_t by taking the derivative:

$$\Delta(x) = -\left.\frac{d}{dt}\right|_{t=0} \phi_t(x).$$

One obtains ρ_α from ϕ_t by Laplace transform:

$$\rho_\alpha = \alpha \int_0^\infty e^{-t\alpha} \phi_t \, dt = \int_0^\infty e^{-t} \phi_{t/\alpha} \, dt.$$

One obtains ϕ_t from ρ_α by considering $\Delta_\alpha = \alpha\Delta(\alpha + \Delta)^{-1} = \alpha(1 - \rho_\alpha)$ and

$$\phi_t = e^{-t\Delta} = \lim_{\alpha \to \infty} e^{-t\Delta_\alpha} \quad \text{(norm-convergent in } \mathbb{B}(L^2M)\text{)}.$$

Theorem 6 (Sauvageot et al.)**.** *Let δ be a real closable derivation with a self-adjoint domain. Then, ϕ_t and ρ_α map M into M and are u.c.p. and τ-symmetric (i.e., $\tau(\phi_t(x^*)y) = \tau(x^*\phi_t(y))$).*

Proof. Note that $x \in L^2M$ is in M and $\|x\|_M \leq 1$ if and only if $|\langle x, y\rangle| \leq 1$ for all $y \in M$ with $\|y\|_1 = \tau(|y|) \leq 1$. Also, $x \geq 0$ if and only if $\langle x, y\rangle \geq 0$ for all $y \in M_+$. Thus, the set of τ-symmetric u.c.p. maps on M is closed in $\mathbb{B}(L^2M)$. Since

$$\phi_t = \lim_{\alpha\to\infty} e^{-t\Delta_\alpha} = \lim e^{-t\alpha} e^{t\alpha\rho_\alpha}$$

and

$$e^{t\alpha\rho_\alpha} = \sum_{n=0}^{\infty} \frac{(t\alpha)^n}{n!}\rho_\alpha^n,$$

complete positivity of ϕ_t follows from that of ρ_α. To show ρ_α's are u.c.p. we may assume $\alpha = 1$ (by scaling δ). Further it suffices to show $\rho_1 = (1+\Delta)^{-1}$ is a positive map on M (by considering $\delta^{(n)} \colon \mathbb{M}_n(M) \to \mathbb{M}_n(\mathcal{H})$).

Let $x \in M$, $0 \leq x \leq 1$ be given and set $y = (1+\Delta)^{-1}(x)$. We will check $0 \leq y \leq 1$. Since $y \in \mathrm{dom}(\Delta) \subset \mathrm{dom}(\delta)$, there is a sequence $z_n \in \mathcal{D}_{\mathrm{sa}}$ such that $\|z_n - y\|_2 \to 0$ and $\|\delta(z_n) - \delta(y)\| \to 0$. Let us show $\|f(z_n) - y\|_2 \to 0$ for $f(t) = 0 \vee t \wedge 1$. For any $z \in \mathcal{D}_{\mathrm{sa}}$ one has (keep $y + \Delta(y) = x$ in mind)

$$\begin{aligned}
\|z-y\|_2^2 + \|\delta(z)-\delta(y)\|^2 &= \|z\|_2^2 - 2\langle z, y\rangle + \|y\|_2^2 + \|\delta(z)\|^2 - 2\langle y, \Delta(y)\rangle + \langle y, \Delta(y)\rangle \\
&= \|z\|_2^2 - 2\langle z, x\rangle + \|\delta(z)\|^2 + \langle y, x\rangle - \|x\|_2^2 + \|x\|_2^2 \\
&= \|z-x\|_2^2 + \|\delta(z)\|^2 - \langle \Delta(y), x\rangle.
\end{aligned}$$

Therefore, by Lemmas 1, 3 and 5, one has

$$\begin{aligned}
\|f(z_n) - y\|_2^2 &\leq \|f(z_n) - x\|_2^2 + \|\delta(f(z_n))\|^2 - \langle \Delta(y), x\rangle \\
&\leq \|z_n - x\|_2^2 + \|\delta(z_n)\|^2 - \langle \Delta(y), x\rangle \\
&= \|z_n - y\|_2^2 + \|\delta(z_n) - \delta(y)\|^2 \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

since $\|f\|_{\mathrm{Lip}} = 1$ and $f(x) = x$. This proves $(1+\Delta)^{-1}$ is positive and bounded. □

We sketch the argument constructing a derivation from a τ-symmetric u.c.p. semigroup ϕ_t. By the Hille-Yoshida Theorem, there is a positive selfadjoint operator Δ on L^2M such that $\phi_t = e^{-t\Delta}$.

One proves that $\mathcal{D} = M \cap \mathrm{dom}(\Delta^{1/2})$ is a weakly dense $*$-subalgebra (proof omitted).

We observe that for all $x, y \in \mathcal{D}$, the derivative of $\tau(\phi_t(x^*)y)$ at 0 exists and is equal to $-\langle \Delta^{1/2}(y), \Delta^{1/2}(x)\rangle$. Indeed, by polarization, we may assume $y = x$. Let $\Delta = \int s\, dE(s)$ be the spectral decomposition and set $E_x(B) = \langle E(B)x, x\rangle$. Then, $\int s\, dE_x(s) = \|\Delta^{1/2}x\|^2 < \infty$. Since $0 \leq 1 - e^{-ts} \leq ts$,

$$\lim_{t\searrow 0} \frac{1}{t}\tau(x^*x - \phi_t(x^*)x) = \lim_{t\searrow 0} \int_0^\infty \frac{1-e^{-ts}}{t}\, dE_x(s) = \int_0^\infty s\, dE_x(s) = \|\Delta^{1/2}x\|^2$$

by Lebesgue's theorem. Hence, we can define a sesquilinear form on $\mathcal{D}\otimes\mathcal{D}$ by

$$\begin{aligned}\langle y\otimes b, x\otimes a\rangle &= \frac{d}{dt}\Big|_{t=0} \tau(a^*(\phi_t(x^*y)-\phi_t(x^*)\phi_t(y))b)\\ &= \lim_{t\to 0}\frac{1}{t}\tau(a^*(\phi_t(x^*y)-\phi_t(x^*)\phi_t(y))b).\end{aligned}$$

It is not difficult to check that this form is positive semi-definite. (Use Stinespring's Dilation Theorem.) We obtain the Hilbert space $\mathcal{H}$ from $\mathcal{D}\otimes\mathcal{D}$ by separation and completion. By a direct computation, one check that

$$J(x\otimes a) = a^*\otimes x^* - a^*x^*\otimes 1$$

defines a conjugate-linear isometric involution on $\mathcal{H}$. (Note that $1\otimes a = 0$ in $\mathcal{H}$.) The right action of $\mathcal{D}$ on $\mathcal{H}$ is given by

$$(x\otimes a)\cdot z = x\otimes az.$$

It is not difficult to check that this extends to a normal opposite $*$-representation on M. The left action of M on $\mathcal{H}$ is given by the right action conjugated by J. If I haven't made any mistake, the left action should be

$$z(x\otimes a) = zx\otimes a - z\otimes xa.$$

Finally, define the derivation $\delta\colon \mathcal{D}\to\mathcal{H}$ by

$$\delta(x) = x\otimes 1$$

and confirm that $\|\delta(x)\|^2 = \|\Delta^{1/2}(x)\|_2^2$.

3. Peterson's Theorem

We collect here notations. Let δ be a real closable derivation,

$$\Delta = \delta^*\delta,\ \zeta_\alpha = \sqrt{\frac{\alpha}{\alpha+\Delta}},\ \tilde{\delta}_\alpha = \alpha^{-1/2}\delta\circ\zeta_\alpha$$

(note that $\operatorname{ran}\zeta_\alpha \subset \operatorname{dom}\Delta^{1/2} = \operatorname{dom}\delta$) and

$$\tilde{\Delta}_\alpha = \alpha^{-1/2}\Delta^{1/2}\circ\zeta_\alpha = \sqrt{\frac{\Delta}{\alpha+\Delta}},\ \theta_\alpha = 1-\tilde{\Delta}_\alpha.$$

Recall also that $\rho_\alpha = \alpha(\alpha+\Delta)^{-1}$ and $\phi_t = \exp(-t\Delta)$. All operators are firstly defined as Hilbert space operators. Since $1-\sqrt{t}\le\sqrt{1-t}$ for all $0\le t\le 1$, one has $\theta_\alpha\le\zeta_\alpha$ and

$$\|a-\zeta_\alpha(a)\|_2 \le \|\tilde{\Delta}_\alpha(a)\|_2 = \|\tilde{\delta}_\alpha(a)\|_2 \le \|a\|_2$$

for all $a\in M$. It is also not too difficult to see that for $\Omega\subset(L^2M)_1$, the four values

$$\|a-\phi_{1/\alpha}(a)\|_2,\ \|a-\eta_\alpha(a)\|_2,\ \|a-\zeta_\alpha(a)\|_2,\ \|\tilde{\delta}_\alpha(a)\|$$

converge to zero uniformly for $a \in \Omega$ if one of them converges to zero uniformly. For instance, since $(\beta/(\alpha+\beta))^{1/2}\|\chi_{[\beta,\infty)}(\Delta)a\|_2 \le \|\tilde{\delta}_\alpha(a)\|$ for every β, taking $\beta = \|\tilde{\delta}_\alpha(a)\|\alpha$, one has

$$\|a - \phi_{1/\alpha}(a)\|_2 \le (1 - \exp(-\|\tilde{\delta}_\alpha(a)\|)\|a\|_2 + \sqrt{\|\tilde{\delta}_\alpha(a)\| + \|\tilde{\delta}_\alpha(a)\|^2}.$$

Lemma 7. *One has*

$$\zeta_\alpha = \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{t}(t+1)}\rho_{\alpha(t+1)/t}\,dt$$

and

$$\tilde{\Delta}_\alpha = \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{t}(t+1)}\big(1 - \rho_{\alpha t/(t+1)}\big)\,dt.$$

In particular, ζ_α and θ_α are u.c.p. τ-symmetric maps on M.

Proof. Since

$$s^{1/2} = \frac{1}{\pi}\int_0^\infty \frac{s}{\sqrt{t}(t+s)}\,dt$$

for any $s \ge 0$, one has

$$\rho_\alpha^{1/2} = \frac{1}{\pi}\int_0^\infty \frac{\rho_\alpha}{\sqrt{t}(t+\rho_\alpha)}\,dt.$$

But $\rho_\alpha = \alpha(\alpha+\Delta)^{-1}$ implies

$$\frac{\rho_\alpha}{t+\rho_\alpha} = \frac{\alpha}{\alpha(t+1)+t\Delta} = \frac{1}{t+1}\rho_{\alpha(t+1)/t}.$$

This proves the first identity. The proof of the second identity is similar. □

Lemma 8. *$\psi_t = e^{-t\Delta^{1/2}}$ is a semigroup of τ-symmetric u.c.p. maps on M.*

Proof. Let $\Delta_\alpha = \alpha\Delta(\alpha+\Delta)^{-1} = \alpha(1-\rho_\alpha)$. Then, $\Delta_\alpha^{1/2} = \alpha^{1/2}(1-\rho_\alpha)^{1/2} = \alpha^{1/2}(1-\theta_\alpha)$ and

$$\psi_t = \lim_{\alpha\to\infty} e^{-t\Delta_\alpha^{1/2}} = \lim_{\alpha\to\infty} e^{-t\alpha^{1/2}}e^{t\alpha^{1/2}\theta_\alpha}.$$

Since θ_α is a u.c.p. map on M, so is ψ_t. □

Lemma 9. *For $x, y \in \mathcal{D}$, one has*

$$\|\Delta^{1/2}(x^*)y + x^*\Delta^{1/2}(y) - \Delta^{1/2}(x^*y)\|_2 \le 4\sqrt{\|\delta(x)\|\|x\|_\infty\|\delta(y)\|\|y\|_\infty}.$$

Proof. For every $x, y \in \mathcal{D}$, one has

$$\begin{aligned}\Gamma(x^*,y) &:= \Delta^{1/2}(x^*)y + x^*\Delta^{1/2}(y) - \Delta^{1/2}(x^*y)\\ &= \frac{d}{dt}\bigg|_{t=0}(\psi_t(x^*y) - \psi_t(x^*)\psi_t(y))\\ &= \lim_{t\to 0}\frac{\psi_t(x^*y) - \psi_t(x^*)\psi_t(y)}{t},\end{aligned}$$

where the limit converges in L^2M. It follows that the sesquilinear form

$$\langle y \otimes b, x \otimes a\rangle = \tau(a^*\Gamma(x^*,y)b)$$

defined on $\mathcal{D} \otimes M$ is positive semi-definite. By Cauchy-Schwarz inequality,

$$\begin{aligned}\|\Gamma(x^*,y)\|_2 &= \sup\{|\tau(a^*\Gamma(x^*,y)b)| : a,b \in M,\ \|aa^*\|_2 \le 1,\ \|bb^*\|_2 \le 1\}\\ &\le \sup\{\|x\otimes a\|\|y\otimes b\| : a,b \in M,\ \|aa^*\|_2 \le 1,\ \|bb^*\|_2 \le 1\}\\ &\le \|\Gamma(x^*,x)\|_2^{1/2}\|\Gamma(y^*,y)\|_2^{1/2}.\end{aligned}$$

Since

$$\|\Delta^{1/2}(x^*x)\|_2 = \|\delta(x^*x)\| \le \|\delta(x^*)x\| + \|x^*\delta(x)\| \le \|\delta(x)\|\|x\|_\infty,$$

one has

$$\|\Gamma(x^*,x)\| \le \|\Delta^{1/2}(x^*)x\|_2 + \|x^*\Delta^{1/2}(x)\|_2 + \|\Delta^{1/2}(x^*x)\|_2 \le 4\|\delta(x)\|\|x\|_\infty.$$

The same for y. □

Theorem 10 (Peterson)**.** *Let δ be a real closable derivation, and ζ_α, $\tilde{\delta}_\alpha$ be defined as above. Then, for every $a, x \in M$, one has*

$$\|\zeta_\alpha(a)\tilde{\delta}_\alpha(x) - \tilde{\delta}_\alpha(ax)\| \le 10\|x\|_\infty\|a\|_\infty^{1/2}\|\tilde{\delta}_\alpha(a)\|^{1/2}$$

and

$$\|\tilde{\delta}_\alpha(x)\zeta_\alpha(a) - \tilde{\delta}_\alpha(xa)\| \le 10\|x\|_\infty\|a\|_\infty^{1/2}\|\tilde{\delta}_\alpha(a)\|^{1/2}.$$

Proof. One has

$$\zeta_\alpha(a)\tilde{\delta}_\alpha(x) = \alpha^{-1/2}\delta(\zeta_\alpha(a)\zeta_\alpha(x)) - \tilde{\delta}_\alpha(a)\zeta_\alpha(x) =: A_1 - A_2.$$

We note that $\|A_2\| \le \|x\|_\infty\|\tilde{\delta}_\alpha(a)\|$. Let $\delta = V\Delta^{1/2}$ be the polar decomposition. Then, one has

$$\begin{aligned}A_1' &:= V^*A_1\\ &= \zeta_\alpha(a)\tilde{\Delta}_\alpha(x) + \tilde{\Delta}_\alpha(a)\zeta_\alpha(x) - \alpha^{-1/2}\Gamma(\zeta_\alpha(a),\zeta_\alpha(x))\\ &=: \qquad B_1 \qquad + \qquad B_2 \qquad - \qquad B_3\end{aligned}$$

in $L^2(M)$. We note that $\|B_2\| = \|A_2\| \le \|x\|_\infty\|\tilde{\delta}_\alpha(a)\|$; and by Lemma 9 that $\|B_3\| \le 4\|x\|_\infty\|a\|_\infty^{1/2}\|\tilde{\delta}_\alpha(a)\|^{1/2}$. Finally, one has

$$B_1 = \zeta_\alpha(a)\tilde{\Delta}_\alpha(x) = \zeta_\alpha(a)(1-\theta_\alpha)(x) \approx ax - \theta_\alpha(ax) = \tilde{\Delta}_\alpha(ax).$$

For the above estimates, we used

$$\|\zeta_\alpha(a)x - ax\|_2 \le \|x\|_\infty\|a - \zeta_\alpha(a)\|_2 \le \|x\|_\infty\|\tilde{\delta}_\alpha(a)\|_2$$

and

$$\begin{aligned}\|\zeta_\alpha(a)\theta_\alpha(x)-\theta_\alpha(ax)\|_2 &\le \|x\|_\infty\|(\zeta_\alpha-\theta_\alpha)(a)\|_2+\|\theta_\alpha(a)\theta_\alpha(x)-\theta_\alpha(ax)\|_2\\ &\le \|x\|_\infty(\|\tilde{\delta}_\alpha(a)\|_2+2\|a\|_\infty^{1/2}\|\tilde{\delta}_\alpha(a)\|^{1/2})\end{aligned}$$

(see Lemma 11 below). Consequently, one has

$$\zeta_\alpha(a)\tilde{\delta}_\alpha(x)\approx A_1\approx VB_1=\tilde{\delta}_\alpha(ax).$$

This yields the first inequality. Since the derivation is real, one gets the second as well. □

Lemma 11. *Let (M,τ) be a finite von Neumann algebra and θ be a τ-symmetric u.c.p. map on M. Then for every $a,x\in M$, one has*

$$\|\theta(ax)-\theta(a)\theta(x)\|_2\le 2\|x\|_\infty\|a\|_\infty^{1/2}\|a-\theta(a)\|_2^{1/2}.$$

Proof. Let $\theta(x)=V^*\pi(x)V$ be a Stinespring dilation. Then,

$$\begin{aligned}\|\theta(ax)-\theta(a)\theta(x)\|_2 &= \|V^*\pi(x)(1-VV^*)\pi(a^*)V\hat{1}\|_2\\ &\le \|x\|_\infty\|(1-VV^*)^{1/2}\pi(a^*)V\hat{1}\|_2\\ &= \|x\|_\infty\tau\big(\theta(aa^*)-\theta(a)\theta(a^*)\big)^{1/2}.\end{aligned}$$

Since $\tau\circ\theta=\tau$, this completes the proof. □

E-mail address: `ozawa@math.ucla.edu`