

Math 175: Vectors (13.2) and the Dot Product (13.3)

1. VECTORS AND ITS BASIC PROPERTIES

(1) A vector is used to indicate a quantity that has both magnitude and direction.

- A vector is often represented by an arrow or a directed line segment. The length represents the magnitude and the arrow points to the direction of the vector.
- Thus two vectors are the same if they have the same magnitude and direction, no matter where is their initial points and terminal points.

(2) How to combining two vectors?

- The sum $\vec{u} + \vec{v}$ of two vectors: we use Triangle Law or Parallelogram Law.
- Scalar multiplication: c is a scalar (a number) and \vec{v} is vector. Then $c\vec{v}$ gives a new vector whose length is $|c|$ times the length of \vec{v} , and direction is the same as \vec{v} (if $c > 0$) or opposite to \vec{v} (if $c < 0$).
- The difference $\vec{u} - \vec{v}$ is defined to $\vec{u} + (-\vec{v})$.

Example: given \vec{a} and \vec{b} , draw $\vec{a} - 2\vec{b}$.

(3) How to study vectors in coordinate system?

(a) In a coordinate system, if we can place the initial points of all vectors at the origin, thus a vector is totally determined by its terminal point, which corresponds to a coordinate, for example (x, y, z) in a 3-D system. On the other hand, every point P in a 3-D system is associated with a position vector \vec{OP} . That is

$$\langle x_1, y_1, z_1 \rangle \leftrightarrow (x_1, y_1, z_1).$$

Given two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, we can also represent vector \vec{AB} :

$$\vec{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle .$$

How to prove it?

(b) When putting vectors in coordinate system, we can easily find their lengths.

Given a vector $\vec{a} = \langle x_1, y_1, z_1 \rangle$, then length of \vec{a} is

$$|\vec{a}| = \sqrt{x_1^2 + y_1^2 + z_1^2}.$$

(c) The sum, scalar multiplication, and difference can also be easily computed.

Given vectors $\vec{a} = \langle x_1, y_1, z_1 \rangle$, $\vec{b} = \langle x_2, y_2, z_2 \rangle$, and a real number c ,

$$\vec{a} + \vec{b} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle, \vec{a} - \vec{b} = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle, c\vec{a} = \langle cx_1, cy_1, cz_1 \rangle .$$

Example: if $\vec{a} = \langle 4, 0, 3 \rangle$ and $\vec{b} = \langle -2, 1, 5 \rangle$, find $|\vec{a}|$ and the vectors $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $3\vec{b}$, and $2\vec{a} + 5\vec{b}$.

Here are some properties of vectors you can easily prove either algebraically or geometrically.

$$\begin{aligned}\vec{a} + \vec{b} &= \vec{b} + \vec{a}, \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}, \vec{a} + \vec{0} = \vec{a}, \vec{a} + (-\vec{a}) = \vec{0} \\ c(\vec{a} + \vec{b}) &= c\vec{a} + c\vec{b}, (c + d)\vec{a} = c\vec{a} + d\vec{a}, (cd)\vec{a} = c(d\vec{a}), 1\vec{a} = \vec{a}\end{aligned}$$

By introducing unit vectors $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$ and $\vec{k} = \langle 0, 0, 1 \rangle$, we can write a vector $\vec{a} = \langle x_1, y_1, z_1 \rangle$ as follows

$$\vec{a} = \langle x_1, y_1, z_1 \rangle = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}.$$

The vectors \vec{i}, \vec{j} and \vec{k} correspond to the unit vectors for x -, y -, and z -axis. Since every vector in V_3 can be expressed in terms of \vec{i}, \vec{j} and \vec{k} , \vec{i}, \vec{j} and \vec{k} are called the standard basis vectors for V_3 .

Example: if $\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{b} = 4\vec{i} + 7\vec{k}$, express the vector $2\vec{a} + 3\vec{b}$ in terms of i, \vec{j} and k .

Each vector \vec{u} also corresponds to a unit vector $\frac{1}{|\vec{u}|}\vec{u}$. (Can you show that it is a really a unit vector?)

Example: find the unit vector in the direction of the vector $2\vec{i} - \vec{j} - 2\vec{k}$.

2. THE DOT PRODUCT OF TWO VECTORS

Two vectors may also give us a useful quantity using so-called dot product.

(1) **Definition:** If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the dot product of \vec{a} and \vec{b} is defined to be

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Examples: $\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle$, $(\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (2\vec{j} - \vec{k})$

(2) **Properties of dot product:**

$$\begin{aligned}\vec{a} \cdot \vec{a} &= |\vec{a}|^2, & \vec{a} \cdot \vec{b} &= \vec{b} \cdot \vec{a}, & \vec{0} \cdot \vec{a} &= 0 \\ \vec{a} \cdot (\vec{b} + \vec{c}) &= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}, & (c\vec{a}) \cdot \vec{b} &= c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})\end{aligned}$$

Can you prove those?

(3) **Dot product and angles**

Using dot product, we can find the angle formed by two vectors.

Let θ be the angle formed by \vec{u} and \vec{v} , then

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta.$$

Can you prove this?

As a result,

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}.$$

Examples: (i) If two vectors have lengths 4 and 6, and the angle between them is $\pi/3$, find the dot product.

(ii) Find the angle between $\vec{a} = \langle 2, 2, -1 \rangle$ and $\vec{b} = \langle 5, -3, 2 \rangle$.

In particular, we can tell when \vec{u} is orthogonal to \vec{v} .

two vectors \vec{u} and \vec{v} are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$

We can also tell when \vec{u} is parallel to \vec{v} : $|\vec{u} \cdot \vec{v}| = |\vec{u}||\vec{v}|$ iff \vec{u} is parallel to \vec{v} .

(4) Dot product and projection

Let α, β, γ be the angles forms with \vec{a} with x -, y -, and z -axis, respectively. These three angles are called the direction angle of \vec{a} . Then

$$\frac{1}{|\vec{a}|} \vec{a} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle .$$

Example: find the direction angle of the vector $\vec{a} = \langle 1, 2, 3 \rangle$.

Given two vectors \vec{a} and \vec{b} , we can consider the vector projection $proj_{\vec{a}} \vec{b}$ of \vec{b} on \vec{a} .

$$proj_{\vec{a}} \vec{b} = (|\vec{b}| \cos \theta) \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}.$$

Note that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta = |\vec{a}|(|\vec{b}| \cos \theta)$. We call $|\vec{b}| \cos \theta$ to be scalar projection of \vec{b} onto \vec{a} . Note that it could be positive or negative.

Example: find the scalar projection and vector projection of $\vec{b} = \langle 1, 1, 2 \rangle$ onto $\vec{a} = \langle -2, 3, 1 \rangle$.

Dot product has a natural interpretation in physics: suppose that force \vec{F} moves an object from P to Q , then the work done is

$$W = \vec{F} \cdot \vec{PQ}.$$

Example: A force is given by a vector $\vec{F} = 3\vec{i} + 4\vec{j} + 5\vec{k}$ and moves a particle from $P(2, 1, 0)$ to $Q(4, 6, 2)$. Find the work done.