Edge decomposition and game coloring number

Gexin Yu
gexin.yu@vanderbilt.edu

Vanderbilt University

Joint work with O. Borodin, A. Kostochka, and N. Sheikh

Chromatic Number and Maximum back degree

- By greedy coloring: $\chi(G) \leq \Delta(G) + 1$.

Chromatic Number and Maximum back degree

- By greedy coloring: $\chi(G) \leq \Delta(G) + 1$.
- Let $v_1, v_2, \ldots, v_n$ be an ordering of $V(G)$. Let $G_i = G[v_1, \ldots, v_i]$ and $d_i = d_{G_i}(v_i)$. The $\max_i d_i$ is called maximum back degree. Then

$$\chi(G) \leq 1 + \max_i d_i.$$

Game chromatic number and game coloring number

- Alice and Bob play a coloring game with a set of colors in alternate turns. Alice tries to color the graph, and Bob tries to prevent this. Alice has the first move. Alice wins if she can color the graph, otherwise Bob wins.

Game chromatic number and game coloring number

- Alice and Bob play a coloring game with a set of colors in alternate turns. Alice tries to color the graph, and Bob tries to prevent this. Alice has the first move. Alice wins if she can color the graph, otherwise Bob wins.
- The minimum number of colors needed for which Alice has a winning strategy is called the game chromatic number, denoted by $\chi_g(G)$.

Game chromatic number and game coloring number

- Alice and Bob play a coloring game with a set of colors in alternate turns. Alice tries to color the graph, and Bob tries to prevent this. Alice has the first move. Alice wins if she can color the graph, otherwise Bob wins.
- The minimum number of colors needed for which Alice has a winning strategy is called the game chromatic number, denoted by $\chi_g(G)$.
- Strategy: color the vertices in an order such that every uncolored vertex has bounded colored neighbors.

Game chromatic number and game coloring number

- Alice and Bob play a coloring game with a set of colors in alternate turns. Alice tries to color the graph, and Bob tries to prevent this. Alice has the first move. Alice wins if she can color the graph, otherwise Bob wins.
- The minimum number of colors needed for which Alice has a winning strategy is called the game chromatic number, denoted by $\chi_g(G)$.
- Strategy: color the vertices in an order such that every uncolored vertex has bounded colored neighbors.
- Analogous to chromatic number, we can use this "game maximum back number" to bound the game chromatic number.

Game chromatic number and game coloring number

- Alice and Bob play a coloring game with a set of colors in alternate turns. Alice tries to color the graph, and Bob tries to prevent this. Alice has the first move. Alice wins if she can color the graph, otherwise Bob wins.
- The minimum number of colors needed for which Alice has a winning strategy is called the game chromatic number, denoted by $\chi_g(G)$.
- Strategy: color the vertices in an order such that every uncolored vertex has bounded colored neighbors.
- Analogous to chromatic number, we can use this "game maximum back number" to bound the game chromatic number.
- The *game coloring number*, denoted by $col_g(G)$, is $1 + \min$ game maximum back number.

Game chromatic number and game coloring number

- Alice and Bob play a coloring game with a set of colors in alternate turns. Alice tries to color the graph, and Bob tries to prevent this. Alice has the first move. Alice wins if she can color the graph, otherwise Bob wins.
- The minimum number of colors needed for which Alice has a winning strategy is called the game chromatic number, denoted by $\chi_g(G)$.
- Strategy: color the vertices in an order such that every uncolored vertex has bounded colored neighbors.
- Analogous to chromatic number, we can use this "game maximum back number" to bound the game chromatic number.
- The *game coloring number*, denoted by $col_g(G)$, is $1 + \min$ game maximum back number.
- Clearly, $\chi_g(G) \leq col_g(G)$.

What is the business of edge-decomposition?

What is the business of edge-decomposition?

- Lemma (Zhu): Let $G = (V, E)$, $G_1 = (V, E_1)$, and $G_2 = (V, E_2)$. If $E = E_1 \cup E_2$, then

$$col_g(G) \leq col_g(G_1) + \Delta(G_2).$$

What is the business of edge-decomposition?

- Lemma (Zhu): Let $G = (V, E)$, $G_1 = (V, E_1)$, and $G_2 = (V, E_2)$. If $E = E_1 \cup E_2$, then

$$col_g(G) \leq col_g(G_1) + \Delta(G_2).$$

- So, a "good" decomposition of edges will give a nice upper bound for $col_g(G)$.

Game coloring number of planar graphs

- Let G be a planar graph.
  Zhu: $col_g(G) \leq 19$. Kierstead: $col_g(G) \leq 18$

Game coloring number of planar graphs

- Let G be a planar graph.
  Zhu: $col_g(G) \leq 19$.   Kierstead: $col_g(G) \leq 18$
- If G is further an outerplanar graph
  Guan and Zhu: $col_g(G) \leq 7$.

Game coloring number of planar graphs

- Let G be a planar graph.
  Zhu: $col_g(G) \leq 19$. Kierstead: $col_g(G) \leq 18$
- If G is further an outerplanar graph
  Guan and Zhu: $col_g(G) \leq 7$.
- When planar graph G has high girth, He, Hou, Lih, Shao, Wang, and Zhu showed that
  (i) if $g(G) \geq 5$, then $col_g(G) \leq 8$;
  (ii) if $g(G) \geq 7$, then $col_g(G) \leq 6$;
  (iii) if $g(G) \geq 11$, then $col_g(G) \leq 5$;
  (iv) if G is C_4-free, then $col_g(G) \leq 11$.

Edge-decompose a planar graph into a matching and a forest

- Theorem (He et al): If a planar graph G has girth at least 11, then $col_g(G) \leq 5$.

Edge-decompose a planar graph into a matching and a forest

- Theorem (He et al): If a planar graph G has girth at least 11, then $col_g(G) \leq 5$.
- Theorem (He et al): Any planar graph with girth at least 11 can be edge-decomposed into a forest and a matching.

Edge-decompose a planar graph into a matching and a forest

- Theorem (He et al): If a planar graph G has girth at least 11, then $col_g(G) \leq 5$.
- Theorem (He et al): Any planar graph with girth at least 11 can be edge-decomposed into a forest and a matching.
- Theorem (Kleitman et al): Any planar graph with girth at least 10 can be edge-decomposed into a forest and a matching.

Edge-decompose a planar graph into a matching and a forest

- Theorem (He et al): If a planar graph G has girth at least 11, then $col_g(G) \leq 5$.
- Theorem (He et al): Any planar graph with girth at least 11 can be edge-decomposed into a forest and a matching.
- Theorem (Kleitman et al): Any planar graph with girth at least 10 can be edge-decomposed into a forest and a matching.
- Theorem (BKSY): Any planar graph with girth at least 9 can be edge-decomposed into a forest and a matching.

Edge-decompose a planar graph into a matching and a forest

- Theorem (He et al): If a planar graph G has girth at least 11, then $col_g(G) \leq 5$.
- Theorem (He et al): Any planar graph with girth at least 11 can be edge-decomposed into a forest and a matching.
- Theorem (Kleitman et al): Any planar graph with girth at least 10 can be edge-decomposed into a forest and a matching.
- Theorem (BKSY): Any planar graph with girth at least 9 can be edge-decomposed into a forest and a matching.
- The smallest girth one can hope is 6.

M-degree and edge-decomposition

- For an edge $e = xy$, let $M(e) = \max\{d(x), d(y)\}$. Let $M^*(G) = \min_e M(e)$.

M-degree and edge-decomposition

- For an edge $e = xy$, let $M(e) = \max\{d(x), d(y)\}$. Let $M^*(G) = \min_e M(e)$.
- (He et al.) If $\delta(G) = 1$ or $M^*(G) \leq k$, then G can be edge-decomposed into a forest and a graph with maximum degree $k - 1$.

M-degree and edge-decomposition

- For an edge $e = xy$, let $M(e) = \max\{d(x), d(y)\}$. Let $M^*(G) = \min_e M(e)$.
- (He et al.) If $\delta(G) = 1$ or $M^*(G) \leq k$, then G can be edge-decomposed into a forest and a graph with maximum degree $k - 1$.
- $M^*(K_{2,n}) = n$.

M-degree and edge-decomposition

- For an edge $e = xy$, let $M(e) = \max\{d(x), d(y)\}$. Let $M^*(G) = \min_e M(e)$.
- (He et al.) If $\delta(G) = 1$ or $M^*(G) \leq k$, then G can be edge-decomposed into a forest and a graph with maximum degree $k - 1$.
- $M^*(K_{2,n}) = n$.
- (Kotzig): $M^*(G) \leq 5$ if G is triangle-free planar graph with $\delta(G) \geq 3$;
- (Borodin): $M^*(G) \leq 10$ if G a planar graph with $\delta(G) \geq 3$;
  $M^*(G) \leq 7$ if G a planar graph with $\delta(G) \geq 4$;
  $M^*(G) \leq 6$ if G a planar graph with $\delta(G) \geq 5$.
  All these bounds are tight.

M-degree and edge-decomposition

- For an edge $e = xy$, let $M(e) = \max\{d(x), d(y)\}$. Let $M^*(G) = \min_e M(e)$.
- (He et al.) If $\delta(G) = 1$ or $M^*(G) \leq k$, then G can be edge-decomposed into a forest and a graph with maximum degree $k - 1$.
- $M^*(K_{2,n}) = n$.
- (Kotzig): $M^*(G) \leq 5$ if G is triangle-free planar graph with $\delta(G) \geq 3$;
- (Borodin): $M^*(G) \leq 10$ if G a planar graph with $\delta(G) \geq 3$; $M^*(G) \leq 7$ if G a planar graph with $\delta(G) \geq 4$; $M^*(G) \leq 6$ if G a planar graph with $\delta(G) \geq 5$. All these bounds are tight.
- Since $K_{2,n}$ contains 4-cycles, what if G is C_4-free?

M-degree in C_4-free planar graphs

M-degree in C_4-free planar graphs

- (He et al.) If G is a connected C_4-free planar graph with $\delta(G) \geq 2$, then $M^*(G) \leq 8$.

M-degree in C_4-free planar graphs

- (He et al.) If G is a connected C_4-free planar graph with $\delta(G) \geq 2$, then $M^*(G) \leq 8$.
- (BKSY) Let G be a graph without 4-cycles and leaves.
  If G is planar or projective-planar, then $M^*(G) \leq 7$;
  If G is embeddable into a torus or the Klein bottle, then $M^*(G) \leq 8$.
  Both bounds are sharp.

M-degree in C_4-free planar graphs

- (He et al.) If G is a connected C_4-free planar graph with $\delta(G) \geq 2$, then $M^*(G) \leq 8$.
- (BKSY) Let G be a graph without 4-cycles and leaves. If G is planar or projective-planar, then $M^*(G) \leq 7$; If G is embeddable into a torus or the Klein bottle, then $M^*(G) \leq 8$. Both bounds are sharp.
- (BKSY) Let $N(S)$ be the Euler characteristic of a surface S. Let G be a C_4-free graph without leaves embedded into a surface S with $N(S) < 0$. If G has more than $-72N(S)$ edges, then $M^*(G) \leq 8$.

Stretch of the proof (Planar case)

Stretch of the proof (Planar case)

- Let the initial charge of $x \in V(G) \cup F(G)$ be $\mu(x) = d(x) - 4$. By Euler's formula,

$$\sum_{x \in V(G) \cup F(G)} (d(x) - 4) = -2.$$

Stretch of the proof (Planar case)

- Let the initial charge of $x \in V(G) \cup F(G)$ be $\mu(x) = d(x) - 4$. By Euler's formula,

$$\sum_{x \in V(G) \cup F(G)} (d(x) - 4) = -2.$$

- Idea: re-distribute (in a smart way) the charges among the vertices and faces (so the total charge remains unchanged), then either $d(x) \geq 0$ for all x, or some reducible structure will appear. This method is called discharging method.

Stretch of the proof (Planar case)

Stretch of the proof (Planar case)

- Let G be a counterexample with fewest vertices. Then $M^*(G) \geq 8$. That is, every edge is incident with a senior vertex (vertex of degree at least 8).

Stretch of the proof (Planar case)

- Let G be a counterexample with fewest vertices. Then $M^*(G) \geq 8$. That is, every edge is incident with a senior vertex (vertex of degree at least 8).
- Claim: for every vertex v, the total number of incident triangular faces and 2-vertices is at most $d(v)$.

Stretch of the proof (Planar case)

- Let G be a counterexample with fewest vertices. Then $M^*(G) \geq 8$. That is, every edge is incident with a senior vertex (vertex of degree at least 8).
- Claim: for every vertex v, the total number of incident triangular faces and 2-vertices is at most $d(v)$.
- Discharging Rules:
  R1) Every non-triangular face gives $1/2$ to each incident vertex of degree 2 or 3;
  R2) Every non-triangular face gives additional $1/2$ to each incident vertex v of degree 2, if the other face incident with v is a triangle.
  R3) Every senior vertex gives $1/2$ to each adjacent 2-vertex and each incident triangular face.

Stretch of the proof (Planar case)

- Let G be a counterexample with fewest vertices. Then $M^*(G) \geq 8$. That is, every edge is incident with a senior vertex (vertex of degree at least 8).
- Claim: for every vertex v, the total number of incident triangular faces and 2-vertices is at most $d(v)$.
- Discharging Rules:
  R1) Every non-triangular face gives $1/2$ to each incident vertex of degree 2 or 3;
  R2) Every non-triangular face gives additional $1/2$ to each incident vertex v of degree 2, if the other face incident with v is a triangle.
  R3) Every senior vertex gives $1/2$ to each adjacent 2-vertex and each incident triangular face.
- Check: $\mu^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. A contradiction.

Open Problems (from He et al)

Open Problems (from He et al)

- Let $\sigma(g)$ be the least integer k such that every planar graph with girth g can be edge-decomposed into a forest and a graph H with $\Delta(H)$. Determine $\sigma(g)$ for $g \geq 5$.

Open Problems (from He et al)

- Let $\sigma(g)$ be the least integer k such that every planar graph with girth g can be edge-decomposed into a forest and a graph H with $\Delta(H)$. Determine $\sigma(g)$ for $g \geq 5$.

$$g = 5, 6, \text{ then } 1 \leq \sigma(g) \leq 4;$$
$$7 \leq g \leq 8,\ 1 \leq \sigma(g) \leq 2;$$
$$g \geq 9,\ \sigma(g) = 1.$$

Open Problems (from He et al)

- Let $\sigma(g)$ be the least integer k such that every planar graph with girth g can be edge-decomposed into a forest and a graph H with $\Delta(H)$. Determine $\sigma(g)$ for $g \geq 5$.

$$g = 5, 6, \text{ then } 1 \leq \sigma(g) \leq 4;$$
$$7 \leq g \leq 8,\ 1 \leq \sigma(g) \leq 2;$$
$$g \geq 9,\ \sigma(g) = 1.$$

- If a planar graph can be edge-decomposed into a forest and a graph H with $\Delta(H) = k$, how small can the girth of the graph be?

Open Problems (from He et al)

- Let $\sigma(g)$ be the least integer k such that every planar graph with girth g can be edge-decomposed into a forest and a graph H with $\Delta(H)$. Determine $\sigma(g)$ for $g \geq 5$.

$$g = 5, 6, \text{ then } 1 \leq \sigma(g) \leq 4;$$
$$7 \leq g \leq 8, \ 1 \leq \sigma(g) \leq 2;$$
$$g \geq 9, \ \sigma(g) = 1.$$

- If a planar graph can be edge-decomposed into a forest and a graph H with $\Delta(H) = k$, how small can the girth of the graph be?

$$k = 1, \text{ then } 6 \leq g_{\min} \leq 9;$$
$$k = 2, 3, \text{ then } 5 \leq g_{\min} \leq 7;$$
$$k \geq 4, \text{ then } g_{\min} = 5.$$

Thank you!