Vector fields

Definition:

(i) $\underline{F}$ is called a **two dimensional vector field** if $\underline{F}$ is a two dimensional vector valued function of two variables. There exists two scalar valued functions (say, P and Q) of two variables, called **component functions of** $\underline{F}$ such that $\underline{F} = P\underline{i} + Q\underline{j}$.

(ii) $\underline{F}$ is called a **three dimensional vector field** if $\underline{F}$ is a three dimensional vector valued function of three variables. There exists three scalar valued functions (say, P, Q and R) of three variables, called **component functions of** $\underline{F}$ such that $\underline{F} = P\underline{i} + Q\underline{j} + R\underline{k}$.

Example: If f is a scalar valued function of two or three variables, then ∇f is a two or three dimensional vector field respectively.

Definition: A vector field $\underline{F}$ is called **conservative** if there exists a scalar valued function f such that $\underline{F} = \nabla f$. In such a case, f is called the **potential function** of $\underline{F}$.

Theorem 1 (Mixed derivative theorem for two dimensional conservative vector fields): If $\underline{F}(x,y) = P(x,y)\underline{i} + Q(x,y)\underline{j}$ is a **conservative vector field** such that P and Q have continuous first-order partial derivative for (x,y) in D, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{in } D.$$

Theorem2 (Converse of Theorem 1): If $\underline{F}(x,y) = P(x,y)\underline{i} + Q(x,y)\underline{j}$ is a vector field for (x,y) in an **open, simply connected region** D such that P and Q have continuous first-order partial derivative and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in D, then $\underline{F}$ is a conservative vector field in D, that is, there exist a scalar valued function f defined over D such that $\underline{F} = \nabla f$.

Line integrals over an oriented curve

Let C be an oriented curve with a starting and an ending point lying in two or three dimensional space.

(I) **Line integrals of a scalar valued function**

Definition: If f is a scalar valued function of two or three variables, then the **line integral of f over C with respect to arc length,** denoted by $\int_C f(x,y)\,ds$ or $\int_C f(x,y,z)\,ds$, is given by

$$\int_C f(x,y)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*, y_i^*\right)\Delta s_i \quad \text{or} \quad \int_C f(x,y,z)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*, y_i^*, z_i^*\right)\Delta s_i$$

if the limit exists where s denotes the arc length on C. Similarly, one can also define the **line integral of f over C with respect to the given variables**, namely,

$$\int_C f(x,y)\,dx \;\&\; \int_C f(x,y)\,dy \qquad \text{or} \qquad \int_C f(x,y,z)\,dx,\ \int_C f(x,y,z)\,dy \;\&\; \int_C f(x,y,z)\,dz.$$

(II) **Line integrals of a vector field**

Definition: If $\underline{F}$ is a two or three dimensional vector field, then the **line integral of $\underline{F}$ over C,** denoted by $\int_C \underline{F}\cdot\underline{T}\,ds$, is given by

$$\int_C \underline{F}\cdot\underline{T}\,ds = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\underline{F}\left(x_i^*, y_i^*\right)\cdot \underline{T}\Big|_{\left(x_i^*, y_i^*\right)}\right]\Delta s_i \quad \text{or} \quad \int_C \underline{F}\cdot\underline{T}\,ds = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\underline{F}\left(x_i^*, y_i^*, z_i^*\right)\cdot \underline{T}\Big|_{\left(x_i^*, y_i^*, z_i^*\right)}\right]\Delta s_i$$

if the limit exists where $\underline{T}$ denotes unit tangent vector on C.

Formula:

(a) **Two dimensional case:** If the oriented curve C is given by $\underline{r}(t) = \langle x(t), y(t)\rangle$ for $a \le t \le b$, f is a scalar valued function of two variables, $\underline{F} = \langle P, Q\rangle$ is a two dimensional vector field, then

$$\int_C f(x,y)\,ds = \int_a^b f(\underline{r}(t))\left|\underline{r}'(t)\right|dt = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$$\int_C f(x,y)\,dx = \int_a^b f(\underline{r}(t))\,x'(t)\,dt = \int_a^b f(x(t), y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(\underline{r}(t))\,y'(t)\,dt = \int_a^b f(x(t), y(t))\,y'(t)\,dt$$

$$\int_C \underline{F}\cdot\underline{T}\,ds = \int_a^b \left[\underline{F}(\underline{r}(t))\cdot\underline{r}'(t)\right]dt = \int_a^b \left[P(x(t), y(t))\,x'(t) + Q(x(t), y(t))\,y'(t)\right]dt$$

(b) **Three dimensional case:** If the oriented curve C is given by $\underline{r}(t)=\langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$, f is a scalar valued function of three variables, $\underline{F}=\langle P, Q, R\rangle$ is a three dimensional vector field, then

$$\int_C f(x,y,z)ds = \int_a^b f(\underline{r}(t))|\underline{r}'(t)|\,dt = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$$

$$\int_C f(x,y,z)dx = \int_a^b f(\underline{r}(t))x'(t)dt = \int_a^b f(x(t),y(t),z(t))x'(t)dt$$

$$\int_C f(x,y,z)dy = \int_a^b f(\underline{r}(t))y'(t)dt = \int_a^b f(x(t),y(t),z(t))y'(t)dt$$

$$\int_C f(x,y,z)dz = \int_a^b f(\underline{r}(t))z'(t)dt = \int_a^b f(x(t),y(t),z(t))z'(t)dt$$

$$\int_C \underline{F}\cdot\underline{T}\,ds = \int_a^b [\underline{F}(\underline{r}(t))\cdot\underline{r}'(t)]dt = \int_a^b [P(x(t),y(t),z(t))x'(t)+Q(x(t),y(t),z(t))y'(t)+R(x(t),y(t),z(t))z'(t)]dt$$

<u>Other Notations</u>: Sometimes $\int_C \underline{F}\cdot\underline{T}\,ds$ is denoted by $\int_C \underline{F}\cdot d\underline{r}$ and also by $\int_C Pdx+Qdy$ or $\int_C Pdx+Qdy+Rdz$.

<u>Remark</u>: The right hand sides in the formulas do not depend on the choice of the representative $\underline{r}$ for the oriented curve $\underline{C}$.

<u>Properties of line integrals</u>:

(i) If $-C$ denote the same curve C but with reverse orientation, then $\int_{-C} f\,ds = \int_C f\,ds$ and $\int_{-C} \underline{F}\cdot\underline{T}\,ds = -\int_C \underline{F}\cdot\underline{T}\,ds$.

(ii) If C_1 and C_2 be two oriented curves such that C_1 finishes at the point where C_2 starts, then $\int_{C_1\cdot C_2} = \int_{C_1} + \int_{C_2}$ where $C_1\cdot C_2$ is the curve obtained by joining C_1 and C_2.

(iii) $\int_C ds = \text{length of the curve } C$.

<u>Fundamental theorem of line integrals</u>: If C is a smooth oriented curve starting at the point A & ending at the point B, and f is a scalar valued function of two or three variables such that ∇f is continuous at all points on C, then

$$\int_C \nabla f\cdot\underline{T}\,ds = f\big|_B - f\big|_A$$