SUBSET SYSTEMS AND GENERALIZED DISTRIBUTIVE LATTICES

By

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Distributive lattices – alone, or with enriched structure – are mathematical objects of fundamental importance. This text studies generalized distributive lattices; the generalization is that certain infinite meets and joins are required to exist. Subset systems (natural rules which select a family of subsets of each poset) j and m label which sets have joins and meets, respectively.

A calculus of subfunctors is developed: using this calculus, it is shown that any subfunctor F of a monad (containing the image of the unit) generates a submonad $\overline{F}$. Under suitable conditions, any partial F-algebra extends to an $\overline{F}$-algebra. The monad $\overline{F}$ for the free distributive (j, m)-complete lattice is the submonad of the completely distributive complete lattice monad generated by a subfunctor obtained from j and m.

The category $\mathbb{DP}^j_m$ of (j, m)-complete lattices which can be embedded in a completely distributive complete lattice is a full subcategory of $\overline{F}$-algebras. $\mathbb{DP}^j_m$ is complete and has coequalizers.

(j, m)-complete families of subsets of a set (generalized topological spaces) are investigated in analogy to classical point-set topology. Assuming suitable restrictions on j and

m, subspaces can be defined. Assuming these restrictions, there are well-behaved categories corresponding to T_0- and sober- spaces.

CHAPTER 1
INTRODUCTION

1.1 Distributive Lattices

A *semilattice* is a set with an operation $\wedge$ satisfying the following universally quantified equations

- $a \wedge a = a$,
- $a \wedge b = b \wedge a$, and
- $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

A *lattice* is a set with two semilattice operations $\wedge$ and $\vee$, which are related by the condition that

$$a \wedge b = a \iff a \vee b = b.$$

A lattice may be partially ordered by defining $a \leq b$ to mean $a \wedge b = a$; with this order $a \wedge b$ is the largest thing smaller than both a and b and $a \vee b$ is the smallest thing larger than both a and b.

A lattice is *distributive* if either of the following, equivalent, universally quantified equations hold

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Let us consider the concept and its relevance.

A distributive lattice bears some resemblance to ordinary arithemetic where $\wedge$ and $\vee$ correspond to addition and multiplication; the principal difficulty with this view is that $a \wedge a = a \vee a = a$, which does not hold in arithmetic. There is more symmetry in the equations defining distributive lattices than in ordinary arithmetic; formally "$\wedge$" and "$\vee$" are interchangable, and switching them reverses the order. Distributive lattices are interesting algebraic structures in the same right as rings – structures with $+$ and $\cdot$ sensibly defined.

Another perspective is that distributive lattices are models of logic, with "$\wedge$" representing "and" and "$\vee$" representing "or." Obviously, the connectives "and" and "or" both satisfy the semilattice rules. In this interpretation, the distributive laws are tautologies and $a \leq b$ means the proposition a implies b. Conventional logics are often described by distributive lattices obeying extra equations, which correspond to additional tautologies to be modeled.

A perspective particularly revelant to the author is that distributive lattices (with some additional structure) describe topological situations. Intuitively, a topological space is an amorphous blob, from which certain pieces can be cleanly removed. The removable pieces are called closed parts and the complements (i.e, things left over after a closed part has been removed) are called open parts. The usual definition of a topological space is a set X, together with a designated family of open subsets, such that

- X and the empty subset are open,
- if U and V are open, so is the intersection $U \cap V$, and
- if (U_i) is a family of open subsets, then the union $\bigcup_i U_i$ is also open.

The lattice of open sets encodes how a space is woven together. Real analysis provides some justification for the usual definition of a topological space. However, the author wondered how the concept of a topology changes if one varies the definition by requiring either fewer unions of open sets be open, or more intersections of open sets be open.

1.2 Subset Systems

A category is an abstract class of objects with structure, and maps (or, homomorphisms) which preserve the relevant structure. The maps in a category allow comparisons between objects. Category theory allows formal comparisons between various theories of algebraic objects: e.g., one can compare the category of all rings with the category of all distributive lattices.

Many categories of partially ordered sets (with additional structure) fit into a simple, general pattern. The objects are partially ordered sets in which certain intentionally distinguished subsets have suprema or infima while the maps are order preserving functions which preserve said infima and suprema. $\mathbb{P}$ denotes the category of all posets and order preserving maps.

The challenge is how one selects subsets which have infima and suprema. We follow Thatcher, Wright and Wagner [31], who introduced the useful (but blandly named) concept of a subset system.

Definition and Remarks 1.2.1. A *subset system* Z, is a rule which assigns a family $Z(A)$ of subsets to any poset A, such that for any order preserving map $f : A \to B$,

$$\{f(S) : S \in Z(A)\} \subseteq Z(B).$$

Z-complete posets are posets A in which each set $S \in Z(A)$ has a supremum. (See Erne [7] for more extensive bibliography regarding subset systems.)

The following examples of subset systems may convey the generality and usefulness of the concept.

1. If κ is a cardinal, we use κ to denote the subset system which selects all subsets with cardinality less than κ. We use ∞ for the functor which places no restriction on cardinality. This is a subset system because for any function $f : A \to B$, $S \subseteq A$, with $|A| < \kappa$ implies $|f(A)| < \kappa$.

2. A subset $S \subseteq A$ is (upward) *directed* if for each $x, y \in S$, there exists $u \in S$ such that $x \leq u$ and $y \leq u$. The rule *dir* which selects all directed subsets of a poset is a subset system: if $f(x), f(y) \in f(S)$ and S is directed, then there is $u \in S$ such that $x \leq u$ and $y \leq u$. Thus $f(u)$ is a common upper bound for $f(x), f(y)$.

3. A subset $S \subseteq A$ is (upward) *compatible* if for any $x, y \in S$ there is $u \in A$ such that $x \leq u$ and $y \leq u$. "Compatible" differs from "directed" because for the former $u \in S$, while for the later we only require $u \in A$. Similar arguments show that *compat*, which selects compatible subsets is a subset system.

4. A subset of $S \subseteq A$ is a chain if $x, y \in S$ implies $x \leq y$ or $y \leq x$. The rule *ch* which selects all chains in a poset is a subset system.

5. We say a subset $S \subseteq A$ is (upward) *self-bounded* if there is $s \in S$ such that for all $x \in S$, $x \leq s$; a self-bounded set contains a maximum element. The rule *sb*, which selects all self-bounded sets, is a subset system. Note that any order preserving map preserves joins of upward self-bounded sets.

6. **A non-example:** An anti-chain is a set of pairwise incomparable elements. The rule *ac* which selects anti-chains is not a subset system, because there is an order preserving surjection $f : D \to 2$, where D is the two-point anti-chain and 2 is the two point chain.

7. **Generating examples:** Let $\mathcal{Q}$ be any class of posets closed under order preserving surjections. One may define a subset system $Z_{\mathcal{Q}}$ by

$$Z_{\mathcal{Q}}(A) = \{S \subseteq A : S \in \mathcal{Q}\}.$$

   This construction shows there is a great multitude of subset systems. The subset system, *compat*, described above, is not generated this way, because one cannot determine if $S \subseteq A$ is compatible merely by looking at the poset S with its induced order.

We use subset systems, which we generically call j and m, to select which subsets have joins and meets, respectively. Now we enumerate some categories of interest in this discussion.

$\mathbb{P}^j_m$ – the category of all (j, m)-complete posets: that is, posets in which j-suprema and m-infima exist and are preserved by all maps.

$\mathbb{DP}_\infty^\infty$ – the category of all completely distributive complete lattices.

$\mathbb{DP}_m^j$ – the full subcategory of $\mathbb{P}_m^j$ containing objects which can be $\mathbb{P}_m^j$-embedded in a completely distributive complete lattice. $\mathbb{DP}_m^j$ is discussed in Section 5.3.

$\mathbb{M}_m^j$ – the full subcategory of $\mathbb{P}_m^j$ containing posets with an $\overline{F}$ structure, where $\overline{F}$ is the monad defined in Section 5.3.

$Sp\mathbb{P}_m^j$ – the full subcategory of $\mathbb{P}_m^j$ containing spatial posets. See Chapter 7.

$\mathbb{S}_m^j$ – the category of generalized spaces with (j, m)-complete families of distinguished subsets: see Chapter 7.

$\mathbb{F}_m^j$ – the full subcategory of $Sp\mathbb{P}_m^j$ containing spatial posets with flat spectrum: A is defined to have *flat spectrum* if the maps $A \longrightarrow 2$ are trivially ordered: see Section 7.5.

1.3 Methods and Results

The discussion of $\mathbb{DP}_m^j$ and $\mathbb{M}_m^j$, which offer generalizations of distributive lattices, uses the language of category theory. A quick introduction occurs in Chapter 2. The crucial notion of a free object is formalized by monads, which are introduced in Chapter 3.

Chapter 4 describes a theory of subfunctors. The class of subfunctors of a functor F bears a strong similarity to the power set lattice of a set X. Given a monad, with functor part T, Meseguer's Lemmas 4.2.1, 4.2.2, 4.2.6, and 4.2.7, show that any subfunctor F of T containing all constants has a "monadic closure," i.e., a smallest submonad $\overline{F}$ of T which exceeds F. (Meseguer's Lemmas were formulated and proved by the author, but the technique is similar to one in Meseguer [25].) Intuitively, TX is the full set of polynomials (in the sense of universal algebra) with variables in X, FX is a natural subset of polynomials, and $\overline{F}X$ is the smallest natural subset of polynomials which is closed under composition and contains FX. An algebra structure for T is a way of evaluating "all polynomials"; a partial algebra structure for F is a way of "evaluating polynomials in F." Under suitable conditions, partial algebras extend to $\overline{F}$-algebras.

In Chapter 5, Meseguer's Lemmas are brought to bear upon monads for free complete semilattices and free completely distributive complete lattices. Given any subset systems j and m, there is a submonad $\overline{F}$ of the free completely distributive complete lattice monad. The category $\mathbb{M}^j_m$ of $\overline{F}$-algebras offers a (somewhat mysterious) generalization of the category of distributive lattices. The subcategory $\mathbb{DP}^j_m$ containing all (j, m)-complete posets which may be embedded in a completely distributive lattice is somewhat easier to understand and still well behaved.

The existence of free objects in $\mathbb{DP}^j_m$ contrasts with the nonexistence of free objects in $\mathbb{P}^\infty_\infty$ [9] and the category of complete Boolean algebras [8, 9]. A fundamental difference between these categories and $\mathbb{DP}^j_m$ is the requirement that joins and meets obey a distributive law.

The power of category theory comes as much from what it ignores as what it examines. Significant conclusions are often obtained without examining the "grubby details" of what is going on. But this innocence of "grubby details" limits the scope of investigation. In the case of this document, several nicely behaved categories – $\mathbb{M}^j_m$, $\mathbb{DP}^j_m$, and $Sp\mathbb{P}^j_m$ – are introduced. For general subset systems j and m, the author does not even know if these categories differ! The end of Section 5.3 – from Corollary 5.3.10 onwards – describes most of the author's knowledge on the relationship between these categories.

Chapter 6 explores congruences, quotients and coequalizers in $\mathbb{P}$, $\mathbb{P}^j_m$ and $\mathbb{DP}^j_m$. Much classical algebra (ring theory, lattice theory, group theory, etc.) is simplified by the fact that any surjection is a regular epimorphism. For the categories introduced here, the situation is not so simple. Example 6.2.5 shows that a j-join preserving surjective image of a j-complete poset need not be j-complete.

The results of Chapter 7 pre-date the other results presented here. Herrlich [12] contains a detailed examination of reflections (and coreflections) in categories of topological spaces. This dissertation aimed to generalize results summarized in Herrlich [12], for (j, m)-spaces. A (j, m)-space consists of an underlying set and a family of "open" subsets which are closed under j-unions and m-intersections. Continuous maps of (j, m)-spaces are functions such that preimages of open sets are open. The initial aim was to find reflections and coreflections of the category of (j, m)-spaces (obeying a T_0-style separation axiom), and study how the existence and properties of reflections and coreflections varied depending upon the subset systems j and m. An obvious prerequisite to such a project is knowledge of factorizations of continuous maps.

The chapter contains a description of (j, m)-subspaces and (j, m)-quotients. In addition, Section 7.5 describes a reflection of (j, m)-spaces that corresponds to the T_1 reflection of topological spaces. Lastly, Section 7.6 describes epicomplete (j, m)-spaces.

CHAPTER 2
PRIMER ON CATEGORIES AND POSETS

The text assumes a familiarity with the theory of sets typically used in mathematical arguments. So familiar constructions – unions, intersections, cartesian products, quotients by equivalence classes, functions, Zorn's Lemma, and transfinite induction – are used without further comment. (See Halmos [10] if this background is needed.) A basic familarity with general topology is helpful.

Also some comfort with category theory is assumed. Namely, the reader can fill in the blanks in the following informal definitions.

- A *category* $\mathcal{A}$ consists of a class of objects $\mathrm{Obj}(\mathcal{A})$ and maps $\mathrm{Map}(\mathcal{A})$, such that each object has an identity map, and there is an associative notion of composition of maps.
- The set of $\mathcal{A}$-maps from A_1 to A_2 is denoted $\mathcal{A}(A_1, A_2)$.
- A *functor* $F : \mathcal{A} \to \mathcal{B}$ assigns each $A \in \mathrm{Obj}(\mathcal{A})$ an object $F(A) \in \mathrm{Obj}(\mathcal{B})$ and each $\mathcal{A}$-map $f : A_1 \to A_2$ a $\mathcal{B}$-map $Ff : F(A_1) \to F(A_2)$. The assignment respects composition and identity arrows.
- If $\mathcal{A}$ is a category, $\mathcal{A}^{op}$ is the category with the same objects as $\mathcal{A}$, but all arrows reversed. For a category theoretic concept C, the *dual* is obtained by applying C to $\mathcal{A}^{op}$.
- A *contravariant functor* $\mathcal{A} \to \mathcal{B}$ is a functor $\mathcal{A} \to \mathcal{B}^{op}$.
- Diagrams are used to display the behavior of a collection of maps; a diagram commutes if any composites with the same domain and codomain are equal. For example, the

diagram

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, "h"] & X \arrow[d, "g"] \\
Y \arrow[r, "i"] & Z
\end{tikzcd}$$

commutes if and only if $gf = ih$.

Recall the following properties of functors:

Definition 2.0.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. For each $A_1, A_2 \in \mathcal{A}$, F gives a function from the hom-set $\mathcal{A}(A_1, A_2)$ into $\mathcal{B}(FA_1, FA_2)$ by

$$(f : A_1 \to A_2) \mapsto (Ff : A_1 \to A_2).$$

If, for each A_1 and A_2 this map is onto, then F is said to be *full.* If, for each A_1 and A_2 this map is one-to-one, then F is said to be *faithful.*

A *full subcategory* of a category $\mathcal{A}$ is a category $\mathcal{B}$ such that $\mathrm{Obj}(\mathcal{B}) \subseteq \mathrm{Obj}(\mathcal{A})$ and all $f : A \to B$ with $A, B \in \mathrm{Obj}(\mathcal{B})$ are $\mathcal{B}$-maps. $\mathcal{B} \subseteq \mathcal{A}$ is full if, and only if, the inclusion functor is full.

Good general references for category theory are MacLane [21], Borceux [6], and Herrlich and Strecker [13]. MacLane [21] gives a concise, high level summary of most category theory and includes a chapter on monads. Herrlich and Strecker [13] is quite user friendly and concretely describes many examples of adjoint functors. Borceux [6] covers a large amount of material; the exposition is clear and very detailed.

2.1 Distinguished Maps

Definition and Remarks 2.1.1. Begin by defining a dual pair of concepts which coincide with the notions “injective” and “surjective” in the category **Set**.

1. A map $f : A_1 \to A_2$ is *epi, a.k.a epic* (in noun form, an *epimorphism*) if whenever g and h are maps $A_2 \to A_3$ such that $gf = hf$, then $g = h$.

2. A map $f : A_2 \to A_3$ is *mono, a.k.a. monic* (in noun form, a *monomorphism*) if whenever g and h are maps $A_1 \to A_2$ and such that $fg = fh$, then $g = h$.

One may verify that a composition of epimorphisms (resp. monomorphisms) is epi (resp. mono). Moreover, if $f = ab$ is epi (resp. mono), then a is also epi (resp. b is also mono).

Definition 2.1.2. A map $f : A \to B$ is an *isomorphism*, if there is $g : B \to A$ such that $\mathrm{id}_A = gf$ and $\mathrm{id}_B = fg$.

In most categories of "sets with structure": a map is mono if and only if it is injective, surjective maps are epi, but epimorphisms may not be surjective.

Example 2.1.3. Consider **tfAb** – the category of torsion-free abelian groups, i.e., abelian groups such that

$$na = 0 \implies a = 0$$

for any natural number n and group element a, together with group homomorphisms. The inclusion $i : \mathbb{Z} \to \mathbb{Q}$ of the integers in the rational numbers is epi, but not onto.

In categories of "sets with relational structure," bijective maps are not necessarily isomorphisms.

Example 2.1.4. Consider **Top** – the category of topological spaces and continuous maps. The identity function $i : \mathbb{R}_d \to \mathbb{R}$ from the reals (with discrete topology) to the reals (with the usual topology), is a continuous bijection. However, the inverse function i^{-1} is not continuous.

Example 2.1.5. Consider $\mathbb{P}$ – the category of partially ordered sets and order preserving maps. Either bijection ϕ from the trivially ordered set with two elements to the chain with two elements is order preserving. But the inverse function ϕ^{-1} is not order preserving.

For further discussion and more examples of epimorphisms and monomorphisms see Herrlich and Strecker [13, Section 6] and Borceux [6, Volume 1, Sections 1.7 and 1.8].

Definition and Remarks 2.1.6. Consider a pair of maps $f, g : A_1 \to A_2$. A map $i : A_0 \to A_1$ *right-identifies* f and g if $fi = gi$. A map $i : A_0 \to A_1$ is called the *equalizer* of f and g if:

(eq1) i right-identifies f and g, and

(eq2) i has the feature that whenever $j : B_0 \to A_1$ right-identifies f and g, there is a unique map $e : B_0 \to A_0$ such that $j = ie$.

The definite article is used for equalizers, because (eq2) implies that if $i : A_0 \to A$ and $i' : A_0' \to A$ are equalizers for f and g, then there is an isomorphism $j : A_0 \to A_0'$ such that $i' = ij$. Notation: $i = \mathrm{eq}(f, g)$.

If there are f and g such that $i : A_0 \to A_1$ is the equalizer of f and g, then i is *regular mono*. Regular monomorphisms are monomorphisms. If f is epi and regular mono, then f is an isomorphism. For proofs of the assertions in this paragraph and a discussion of examples, see Borceux [6, Volume 1, Section 2] or Herrlich and Strecker [13, Section 16].

The definitions of "coequalizer" and "regular epi" are dual to "equalizer" and "regular mono," but are repeated for emphasis. $e : A_2 \to A_3$ *left-identifies* f and g if $ef = eg$. A map $e : A_2 \to A_3$ is the coequalizer of f and g if

(coeq1) e left-identifies f and g, and

(coeq2) e has the feature that whenever $d : A_2 \to B_3$ left-identifies f and g, there is a unique map $c : A_3 \to B_3$ such that $e = cd$.

Notation: $e = \mathrm{coeq}(f, g)$. If e is the coequalizer of some pair of maps, then e is called *regular epi*. The duals of all basic properties of regular monomorphisms hold for regular epimorphisms.

Definition 2.1.7. An epimorphism f is *extremal* if whenever $f = gh$ and g is mono, then g is an isomorphism. Dually, a monomorphism f is *extremal* if whenever $f = gh$ and h is epi, then h is an isomorphism. For more detailed discussions, see Borceux [6, Volume 1, Section 4.3] and Herrlich and Strecker [13, Section 17].

Definition 2.1.8. The map $f : A \to B$ is *split mono* if there exists $g : B \to A$ such that $\mathrm{id}_A = gf$. The map $g : B \to A$ is *split epi* if there exists $f : A \to B$ such that $\mathrm{id}_A = gf$. For more information, see Herrlich and Strecker [13, Section 5]. Note that our terminology differs slightly from the reference; "section" and "split mono" are synonyms, as are "retraction" and "split epi."

Lemma 2.1.9. *For A either "mono" or "epi," consider the following statements.*

1. *f is split-A.*

2. *f is regular-A.*

3. *f is extremal-A.*

4. *f is A.*

The implications $1 \implies 2 \implies 3 \implies 4$ *always hold. None of the converses generally hold.* For proof see: ($1 \implies 2$) Herrlich and Strecker [13, 16.15], ($2 \implies 3$) Herrlich and Strecker [13, 17.11] or Borceux [6, Volume 1, 4.3.3(1)], ($3 \implies 4$) holds by definition.

2.2 Bounds

Recall that a *partial order* on a set A is a relation $\leq$ satisfying:

(po1) For all $a \in A$, $a \leq a$.

(po2) Whenever $a \leq b$ and $b \leq c$, $a \leq c$.

(po3) Whenever $a \leq b$ and $b \leq a$, $a = b$.

A *preorder* is a relation that satisfies (po1) and (po2). If $\preceq$ is a preorder on A, define an equivalence relation $\simeq$ on A by

$$a \simeq b \iff a \preceq b \text{ and } b \preceq a.$$

The relation $\preceq$ is a partial order on $A/\simeq$. A set A with a partial order (resp. preorder) is called a *partially ordered set* (resp. *preordered set*).

Definition and Remarks 2.2.1. Let A be a preordered set. Define the up- and down-closures of $x \in A$ by

$$\downarrow x = \{a \in A : a \preceq x\}$$

and

$$\uparrow x = \{a \in A : x \preceq a\}.$$

More generally, if $S \subseteq A$ define

$$\downarrow S = \cup\{\downarrow x : x \in S\} = \{a \in A : \exists s \in S\,,\, a \preceq s\}$$

and

$$\uparrow S = \cup\{\uparrow x : x \in S\} = \{a \in A : \exists s \in S\,,\, s \preceq a\}.$$

If $S \subseteq A$ an *upper (resp. lower) bound* for S is $a \in A$ such that for all $s \in S$, $s \preceq a$ (resp. $a \preceq s$). Use the following notation for the set of upper bounds of S,

$$\overline{\mathbb{B}}(S) = \cap\{\uparrow x : x \in S\} = \{a \in A : \forall s \in S\,,\, s \preceq a\}$$

and similar notation for the set of lower bounds

$$\underline{\mathbb{B}}(S) = \cap\{\downarrow x : x \in S\} = \{a \in A : \forall s \in S\,,\, a \preceq s\}.$$

If $a \in A, S \subseteq A$ and $\uparrow a = \overline{\mathbb{B}}(S)$, then a is a *least upper bound, a.k.a. join, a.k.a. supremum* of S. An equivalent way to say this is

$$a \leq x \iff \forall\, s \in S,\, s \leq x.$$

If both a and a' are suprema of S, then $a, a' \in \overline{\mathbb{B}}(S)$. Thus $a \simeq a'$. In a partially ordered set, the (unique) supremum of S is written $\bigvee S$. If $a \in A, S \subseteq A$ and $\downarrow a = \underline{\mathbb{B}}(S)$, then a is a *greatest lower bound, a.k.a. meet, a.k.a. infimum* of S. An equivalent way to say a is a least upper bound is

$$x \leq a \iff \forall\, s \in S,\, x \leq s.$$

In a partially ordered set, the (unique) infimum of S is denoted $\bigwedge S$. (In using this notion with a preordered set $(A, \preceq)$, one refers to the associated poset $A/\simeq$. For example, $\bigvee S$ then denotes the equivalence class containing all suprema of S.) If $a = \bigvee S$ or $a = \bigwedge S$, a is an *optimum bound* for S.

The following are elementary properties of the bound-operators and optimum bounds.

Lemma 2.2.2. *For any preordered set A (in particular any poset), the following properties hold. Use $\mathbb{B}$ to denote either operator $\overline{\mathbb{B}}$ or $\underline{\mathbb{B}}$.*

1. *For $x, y \in A$, $x \preceq y \iff \downarrow x \subseteq \downarrow y \iff \uparrow y \subseteq \uparrow x$.*
2. *For $S, T \subseteq A$, $S \subseteq T \implies \mathbb{B}(T) \subseteq \mathbb{B}(S)$.*
3. *For $S \subseteq A$, $\overline{\mathbb{B}}(S) = \overline{\mathbb{B}}(\downarrow S)$, $\underline{\mathbb{B}}(S) = \underline{\mathbb{B}}(\uparrow S)$.*
4. *For $(S_i)_{i \in I}$ a family of subsets of A, $\mathbb{B}(\cup S_i) = \cap \mathbb{B}(S_i)$.*
5. *If $f : A \to B$ is order preserving, then $f(\mathbb{B}_A(S)) \subseteq \mathbb{B}_B(f(S))$.*

Definition and Remarks 2.2.3. Let A and B be preordered sets. A *Galois connection* between A and B is a pair of functions $f : A \to B$ and $g : B \to A$ such that

(gc1) f and g are order reversing.

(gc2) For all $a \in A$ and $b \in B$, $a \preceq g(f(a))$ and $b \preceq f(g(b))$.

Below, basic properties of Galois connections are listed. Symmetry in the definition allows symmetry in proofs. For any true statement about Galois connections, then the statement obtained by switching the roles of f and g, along with A and B is also a true statement about Galois connections.

Note that f and g respect $\simeq$. The notation $f \simeq g$ means "for all $a \in A$, $f(a) \simeq g(b)$."

1. $g \simeq gfg$: Suppose $a \in A$. By (gc2), $a \preceq f(g(a))$ and $g(a) \preceq g(f(g(a)))$. Using $a \preceq f(g(a))$ and (gc1), $g(f(g(a))) \preceq g(a)$.

2. $f(A/\simeq)$ is dually order isomorphic to $\{b \in B : b \simeq fg(b)\}$: This follows from the preceding statement. Since $f \simeq fgf$, any member $f(a) \in f(A/\simeq)$ is equivalent to $f(g(f(a)))$. Moreover, if $b \in B$ and $b \simeq f(g(b))$, then $b \in f(A/\simeq)$ because $b \simeq f(g(b))$.

3. $a \preceq g(b) \iff b \preceq f(a)$: if $a \preceq g(b)$, then $b \preceq f(g(b)) \preceq f(a)$. The converse is proved similarly.

4. For any $S \subseteq A$, $\underline{\mathbb{B}}f(S) = g^{-1}\overline{\mathbb{B}}(S)$. Calculate

$$\begin{aligned} x \in \underline{\mathbb{B}}(f(S)) &\iff \forall s \in S, x \preceq f(s) \\ &\iff \forall s \in S, s \preceq g(x) \\ &\iff g(x) \in \overline{\mathbb{B}}(S) \\ &\iff x \in g^{-1}\overline{\mathbb{B}}(S) \end{aligned}$$

5. If A and B are posets, the previous item implies $\bigwedge f(S) = f(\bigvee S)$.

It is helpful to rephrase condition 3 for posets: "$g(b)$ is the largest a with $b \leq f(a)$." In symbols,

$$g(b) = \bigvee\{a \in A : b \preceq f(a)\}.$$

Basic information about Galois connections has been well-known sincethe 1940s; Raney [28] contains a bibliography of this early literature. The basic properties and definition are listed in Herrlich and Strecker [13, Exercise 27Q]. The particular summary here is by the author.

The concept of a Galois connection is symmetric, and allows one to transfer a great deal of information between preordered sets. However, the fact that the functions involved are order reversing is sometimes inconvenient. The concept of an *adjoint connection* between posets is obtained by formally reversing one of the posets involved.

Definition and Remarks 2.2.4. Suppose A and B are preordered sets. A pair $f : A \to B$, $g : B \to A$ of order preserving functions is an *adjoint connection* between A and B if

(ad1) For all $a \in A$, $a \preceq g(f(a))$.

(ad2) For all $b \in B$, $f(g(b)) \preceq b$.

Each basic property for Galois connections corresponds to a basic property of adjoint connections. The basic properties of adjoint connections are listed below; proofs are omitted.

1. $f \simeq fgf$ and $g \simeq gfg$
2. $f(A/\simeq) = \{b \in B : b \simeq g(f(b))\}$ and $g(B/\simeq) = \{a \in A : a \simeq f(g(a))\}$.
3. $f(a) \preceq b \iff a \preceq g(b)$
4. For any $S \subseteq A$, $\underline{\mathbb{B}}g(S) = f^{-1}\underline{\mathbb{B}}(S)$ and $\overline{\mathbb{B}}f(S) = g^{-1}\overline{\mathbb{B}}(S)$
5. If A and B are posets, $f(\bigwedge S) = \bigwedge f(S)$ and $g(\bigvee S) = \bigvee g(S)$.

Because of the asymmetry between f and g, and property 3, f is called the left adjoint and g the right adjoint. Again there is an interpretation of 3 in words. "$f(a)$ is the smallest b such that $a \preceq g(b)$; $g(b)$ is the largest a such that $f(a) \preceq b$."

The definition and basic properties of adjoint connections are "folklore." For another discussion of them see Johnstone [16, Chapter I, Paragraph 3].

Definition 2.2.5. A poset A is *complete* if each subset $S \subseteq A$ (including the empty set) has a supremum. Since the supremum of set of lower bounds for S is a lower bound for S, A is complete if and only if each subset $S \subseteq A$ (including the empty set) has an infimum.

There is a criterion for testing when a given order preserving (resp. reversing) map between posets is part of an adjoint (resp. Galois) connection.

Theorem 2.2.6. Adjoint Existence *Suppose A and B are posets.*

1. *Suppose $f : A \to B$ is order preserving and A is complete. Then f is a left adjoint if and only if $f(\bigvee S) = \bigvee f(S)$ for all $S \subseteq A$.*

2. *Suppose $g : A \to B$ is order preserving and A is complete. Then g is a right adjoint if and only if $g(\bigwedge S) = \bigwedge g(S)$ for all $S \subseteq A$.*

3. *Suppose $f : A \to B$ is order reversing and A is complete. Then f is part of a Galois connection if and only if $f(\bigvee S) = \bigwedge f(S)$ for all $S \subseteq A$.*

This theorem is the poset version of the adjoint functor theorem. Folklore: see Johnstone [16, Chapter I, Section 4, Paragraph 2] or Joyal and Tierney [17, Chapter 1, Section 1].

Proof. A proof for 1 follows; the other items are similar. Define $g : B \to A$ by

$$g(b) = \bigvee \{a \in A : f(a) \leq b\}.$$

Since f preserves suprema, $f(g(b)) \leq b$. Thus, $g(b)$ is the largest a such that $f(a) \leq b$. ☐

In adjoint (resp. Galois) connections, $f(a)$ and $g(b)$ can be defined as suprema or infima. There is a sort of converse to this; one can view the supremum as an adjoint to a particular map. See Lemma 5.1.1.

2.3 Natural Transformations

Maps compare objects in a category, functors compare categories, and natural transformations compare functors. This section contains no new results; results and some exposition are paraphrased from Herrlich and Strecker [13, Section 13].

Definition 2.3.1. Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A natural transformation $\alpha : F \to G$ is a rule which assigns a map $\alpha A : FA \to GA$ to each $A \in \mathcal{A}$ such that if $f : A \to B$ is an $\mathcal{A}$-map the following diagram commutes.

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\eta A"] & FB \arrow[d, "\eta B"] \\
GA \arrow[r, "Gf"] & GB
\end{tikzcd}$$

Construction 2.3.2. *Let $F, G, H : \mathcal{X} \to \mathcal{A}$ be functors, $\alpha : F \to G$ and $\beta : G \to H$ natural transformations. (The situation is drawn in the diagram below.)*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, bend left=60, "F", ""{name=U, below}] \arrow[r, "G"{name=M}] \arrow[r, bend right=60, "H"', ""{name=D, above}] & \mathcal{A}
\arrow[from=U, to=M, shorten=2pt, "\alpha"]
\arrow[from=M, to=D, shorten=2pt, "\beta"]
\end{tikzcd}$$

Then the assignment $(\beta\alpha)A = (\beta A)(\alpha A)$ is a natural transformation.

Proof. Let $f : A \to B$ be a map. Since α and β are natural, each square below commutes.

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\alpha"] & FB \arrow[d, "\alpha"] \\
GA \arrow[r, "Gf"] \arrow[d, "\beta"] & GB \arrow[d, "\beta"] \\
HA \arrow[r, "Hf"] & HB
\end{tikzcd}$$

Therefore, the outside rectangle commutes; hence, $\beta\alpha$ is natural. ☐

Call $\beta\alpha$ the *vertical composition* of α and β.

Construction 2.3.3. *If $F : \mathcal{A} \to \mathcal{B}$ and $G, H : \mathcal{B} \to \mathcal{C}$ are functors and $\alpha : G \to H$ is a natural transformation, then $(\alpha F)A := \alpha(FA)$ is a natural transformation $\alpha F : GF \to HF$.*

Construction 2.3.4. *If $F : \mathcal{B} \to \mathcal{C}$ and $G, H : \mathcal{A} \to \mathcal{B}$ are functors and $\alpha : G \to H$ is a natural transformation, then $(F\alpha)A := F(\alpha A)$ is a natural transformation $F\alpha : FG \to FH$.*

Construction 2.3.5. *Suppose $F, G : \mathcal{A} \to \mathcal{B}$ and $H, J : \mathcal{B} \to \mathcal{C}$ are functors, and $\alpha : F \to G$, $\beta : H \to J$ are natural transformations. (The situation is drawn below.)*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=60, "F", ""{name=U1, below}] \arrow[r, bend right=60, "G"', ""{name=D1, above}] & \mathcal{B} \arrow[r, bend left=60, "H", ""{name=U2, below}] \arrow[r, bend right=60, "J"', ""{name=D2, above}] & \mathcal{C}
\arrow[from=U1, to=D1, "\alpha"]
\arrow[from=U2, to=D2, "\beta"]
\end{tikzcd}$$

Then for each object $A \in \mathrm{Obj}(\mathcal{A})$ *the following square commutes.*

$$\begin{tikzcd}
FHA \arrow[r, "F\beta A"] \arrow[d, "\alpha HA"] & FJA \arrow[d, "\alpha JA"] \\
GHA \arrow[r, "G\beta A"] & GJA
\end{tikzcd}$$

The assignment $(\beta \cdot \alpha)A := (\alpha JA)(F\beta A) = (G\beta A)(\alpha HA)$ *is a natural transformation.*

Proof. The square commutes because α is a natural transformation; to see this, one applies the natural property at the map $\beta A : HA \to JA$. To prove the natural property of $\beta\alpha$, we use the squares which define $\beta\alpha A$ and $\beta\alpha B$. Compare the corners of the squares using maps obtained from f by application of the functors FH, GH, FJ, and GJ. The resulting commutative cube shows $\beta\alpha$ is a natural transformation. ☐

Call $\beta \cdot \alpha$ the *horizontal composition* of α and β.

Definition 2.3.6. A *natural equivalence* $\alpha : F \to G$ is a natural transformation such that each component αA is an isomorphism. Given functors F and G are *naturally equivalent* if there is a natural equivalence $\alpha : F \to G$.

Categories $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if there are functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ such that FG is naturally equivalent to id_B and GF is naturally equivalent to id_A.

$\mathcal{A}$ and $\mathcal{B}$ are *dual* if there exist **contravariant** functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ such that FG is naturally equivalent to id_B and GF is naturally equivalent to id_A.

2.4 (Co)Limits

For intuition, it is useful to view the objects of a category as a preordered class, with the preorder $A \preceq B$ if and only if there is a map $f : A \to B$. Categories are more complex than preordered classes, because there may be many maps f which manifest $A \preceq B$.

Definition and Remarks 2.4.1. Let $\mathcal{A}$ be a category. A *diagram* D in $\mathcal{A}$ (a.k.a. a *small subcategory* of $\mathcal{A}$) is *a set* of objects $\mathrm{Obj}(D) \subseteq \mathrm{Obj}(\mathcal{A})$ and maps $\mathrm{Map}(D) \subseteq \mathrm{Map}(\mathcal{A})$

between them; for technical reasons one requires that if $f, g \in \mathrm{Map}(D)$ and fg is defined then $fg \in \mathrm{Map}(D)$, and that for all $A \in \mathrm{Obj}(D)$, $\mathrm{id}_A \in \mathrm{Map}(D)$. This definition of a diagram is equivalent to, but differs from the one in the majority of the literature; see MacLane [21], Borceux [6], or Herrlich and Strecker [13] for the standard definition.

Suppose D is a diagram. A *source* $(S, \{s_A : S \to A : A \in \mathrm{Obj}(D)\})$ for D consists of $S \in \mathrm{Obj}(\mathcal{A})$ and maps $s_A : S \to A$ such that if $f : A \to A'$ is a map in D, $fs_A = s_{A'}$. A source for D is a "lower bound" compatible with all maps in D. A source $(L, \{\ell_A : A \in \mathrm{Obj}(D)\})$ for D is the *limit* of D if whenever $(S, \{s_A : A \in \mathrm{Obj}(D)\})$ is a source, there is a unique map $c : S \to L$ such that for each $A \in \mathrm{Obj}(D)$, $s_A = \ell_A c$.

Dually, define a *sink, or cosource* for D to be an object S together with maps $i_A : A \to S$ (for $A \in \mathrm{Obj}(D)$) such that for each $f : A \to A'$ in $\mathrm{Map}(D)$, $i_A = i_{A'} f$.

A *colimit* $(C, \{\mu_A : A \in \mathrm{Obj}(D)\})$ for D is a sink for D, such that if $(S, \{i_A : A \in \mathrm{Obj}(D)\})$ is any sink there is a unique map $c : C \to S$ such that for each $A \in \mathrm{Obj}(D)$, $\mu_A = ci_A$.

Note that limits and colimits of D are unique up to compatible isomorphism. To prove this (for limits), suppose (L, ℓ_A) and (L', ℓ'_A) are both limits for D. The limit property guarantees that there are unique compatible maps $c : L \to L'$ and $d : L' \to L$. But $cd : L' \to L'$ and $dc : L \to L$ are both compatible maps. By uniqueness, $cd = \mathrm{id}'_L$ and $dc = \mathrm{id}_L$. From now on, we shall use the definite article when writing about limits and colimits.

Example 2.4.2. Let us consider **Set**. If $\mathcal{X}$ is a set of sets, form a diagram containing all members of $\mathcal{X}$ and no functions. The limit for this diagram is the Cartesian product $(\prod \mathcal{X}, \pi_X)$.

Recall that $\prod \mathcal{X}$ contains all functions $f : \mathcal{X} \to \bigcup \mathcal{X}$,with the feature that $f(X) \in X$ for all $X \in \mathcal{X}$. Such f are called choice functions, because they choose one member of each $X \in \mathcal{X}$. The functions $\pi_X : \prod \mathcal{X} \to X$ are defined by $\pi_X(f) = f(X)$.

If there is a source $(S, \{s_X : S \to X : X \in \mathcal{X}\})$, there is a unique function $c : S \to \prod \mathcal{X}$ making

$$\begin{tikzcd}
\prod \mathcal{X} \arrow[d, "\pi_X"] & S \arrow[l, "c"'] \arrow[dl, "s_X"] \\
X &
\end{tikzcd}$$

commute for each X. It is defined by $c(a)(X) = s_X(a)$.

Motivated by this example, one defines the category-theoretic *product* of a set $\mathcal{X}$ of objects (in any category) as the limit of the diagram containing all members of $\mathcal{X}$ and no maps. In most categories of "sets with structure" products (exist and) look like products in **Set**, with suitable structure added.

In **Set**, the colimit of the diagram containing all members of $\mathcal{X}$ and no functions, is the disjoint union $\coprod \mathcal{X}$, with inclusion maps $\mu_X : X \to \coprod \mathcal{X}$. The colimit property is satisfied by $(\coprod \mathcal{X}, \mu_X)$ because if there is a sink (C, i_X), the function $c : \coprod \mathcal{X} \to C$ – defined by $c(x) = i_X(x)$ for the unique $X \in \mathcal{X}$ containing x – is the unique compatible map.

In categories other than **Set**, coproducts are defined identically. Usually the coproduct of a set $\mathcal{X}$ of $\mathcal{A}$-objects is "the $\mathcal{A}$-object freely generated by $\coprod \mathcal{X}$."

Example 2.4.3. The [co]equalizer of $f, g : A \to B$ is the [co]limit of the diagram containing objects A and B along with maps f and g. The notation for equalizers is customarily simplified by omitting the source map to B. Previously, $i : E \to A$ was defined to be the equalizer if $fi = gi$ and i factors through any other map which right identifies f and g. i is the source map to A. The source map to B is redundant: it must be $fi = gi : E \to B$. Similar notational economy is applied to coequalizers.

Example 2.4.4. Let us consider the diagram

$$\begin{tikzcd}
 & A \arrow[d, "f"] \\
B \arrow[r, "g"] & C.
\end{tikzcd}$$

(The diagram also contains identities for all objects, but for brevity these are omitted.) The limit $(B \times_C A, \pi_A, \pi_B)$ of this diagram is called the pullback of f along g, or the pullback of g along f. To be explicit, $f\pi_A = \pi_B g$ and if $(Q, q_A : Q \to A, q_B : Q \to B)$ satisfies $fq_A = q_B g$, then there is a unique map $i : Q \to B \times_C A$.

In **Set**,

$$B \times_C A = \{(b, c) \in B \times A : g(b) = f(c)\},$$

and the projection maps π are the restrictions of the projections from the cartesian product. Pullbacks in any concrete category $\mathcal{A}$ – equipped with a limit preserving faithful functor $U : \mathcal{A} \to$ **Set** – are computed identically. Two particular instances of pullbacks deserve special attention.

First, let $f : A \to C$ be any map and $g : S \to C$ be a subset inclusion of $S \subseteq C$. Note that $(s, a) \in S \times_C A$ if and only if $g(s) = f(a)$; suppressing mention of g, this reads $S \times_C A = \{(s, b) : f(b) = s\}$. Thus, this pullback is canonically isomorphic to the preimage of S under f. This example partially motivates the name "pullback."

Second, let $f : A \to B$ be any map and consider the pullback of the diagram,

$$\begin{tikzcd}
 & A \arrow[d, "f"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

i.e., the pullback of f along itself. Using the computation for pullbacks in **Set**, given above,

$$A \times_B A = \{(a, a') \in A \times A : f(a) = f(a')\}.$$

This relation on A is often called the kernel of f. Thus, one calls $(A \times_B A, \pi_{left}, \pi_{right})$ the *kernel pair* of f. More in-depth discussion of pullbacks is given in Borceux [6, Volume 1, Section 2.5] and Herrlich and Strecker [13, Section 21].

Definition and Remarks 2.4.5. A category is said to be *complete* if each diagram has a limit; it is said to be *cocomplete* if each diagram has a colimit.

Unlike the situation for posets, a category may be complete without being cocomplete and vice versa. (See Herrlich and Strecher [13, Section 23, p161ff] for a detailed discussion of this and related issues.)

Consider a functor $F : \mathcal{A} \to \mathcal{B}$. If D is a diagram in $\mathcal{A}$, there is a diagram FD with $\mathrm{Obj}(FD) = \{FA : A \in \mathrm{Obj}(D)\}$ and $\mathrm{Map}(FD) = \{Ff : f \in \mathrm{Map}(D)\}$.

Let D be a diagram in $\mathcal{A}$. A functor $F : \mathcal{A} \to \mathcal{B}$ *preserves the limit of* D, if whenever the limit (L, ℓ_A) exists in $\mathcal{A}$, $(DL, D(\ell_A))$ is the limit of FD. F *preserves limits* if for any diagram D, F preserves the limit of D.

If $G : \mathcal{A} \to \mathcal{B}$ is contravariant, D is a diagram in $\mathcal{A}$, and $(L, \{\ell_A : A \in \mathrm{Obj}(D)\})$ is the limit of D, then $(GL, \{G\ell_A : GA \to GL\})$ is a sink for GD. If $(GL, \{G\ell_A : GA \to GL\})$ is the colimit for GD, then G *takes limits to colimits.* Analogously, may G *take colimits to limits.*

2.5 Adjoint Functors

There are several useful concepts of adjoint connections between categories. There is a strong analogy between preordered sets and categories; any category may be preordered by

$$A \preceq B \iff \exists f : A \to B.$$

Categories are more complex, because many maps could manifest $A \preceq B$. We begin with the concept analogous to 2.2.3.

Definition and Remarks 2.5.1. A *(functorial) Galois connection* between categories $\mathcal{A}$ and $\mathcal{B}$ consists of contravariant functors $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$, together with natural transformations

$$\eta : \mathrm{id}_{\mathcal{A}} \to GF$$

and

$$\epsilon : \mathrm{id}_{\mathcal{B}} \to FG$$

such that $(F\eta A)(\epsilon FA) = \mathrm{id}_{FA}$ and $(G\epsilon B)(\eta GB) = \mathrm{id}_{GB}$ for each $A \in \mathrm{Obj}(\mathcal{A})$ and $B \in \mathrm{Obj}(\mathcal{B})$; these equations are the so-called triangle identities. (Alternate terminology: if (F, G, η, ϵ) is a functorial Galois connection, then F and G are adjoint on the right.)

1. The categories $\mathrm{fix}(\eta)$ and $\mathrm{fix}(\epsilon)$ – containing all objects such that ηA (resp ϵB) is an isomorphism – are dual. [3, Section 4, Lemma 1]

2. There is a natural bijection $\mathcal{A}(A, GB) \to \mathcal{B}(B, FA)$ given by

   $$(f : A \to GB) \mapsto F(f)(\epsilon B) : B \to FA.$$

   The inverse map is

   $$(g : B \to FA) \mapsto G(f)(\eta A) : A \to GB.$$

   There are two (identical) calculations required to check that the functions are mutually inverse. One is summarized by the diagram below.

   $$\begin{tikzcd}
   GFA \arrow[r, "(GFf)"] & GFGB \arrow[r, "G\epsilon B"] & GB \\
   A \arrow[u, "\eta A"] \arrow[r, "f"] & GB \arrow[u, "\eta GB"] \arrow[ur, equal] &
   \end{tikzcd}$$

   The square commutes because η is natural. The triangle commutes because of the identity $\mathrm{id}_{GB} = (G\epsilon B)(\eta GB)$. The reader may formulate and check what is meant by “naturality” of the bijection.

3. F and G both take colimits to limits.

4. Each ηA has the following universal property: if $f : A \to GB$, there is a unique map $\overline{f} = F(f)(\epsilon B) : B \to FA$ which makes the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "\eta A"] \arrow[dr, "f"'] & GFA \arrow[d, "G\overline{f}"] \\
 & GB
\end{tikzcd}$$

   Each ϵB has the analogous universal property.

Much of the literature just deals with adjoint connections, where both functors are covariant. By duality, any such result can be translated in terms of Galois connections. Banaschewski and Bruns [3] includes a reasonably thorough expository section on functors which are adjoint on the right.

The "(functorial) Galois connection" concept is symmetric, but the contravariance of the functors involved is sometimes awkward. An analogous asymmetric concept, with the functors both covariant is described below.

Definition and Remarks 2.5.2. A *(functorial) adjoint connection* between catego-ries $\mathcal{A}$ and $\mathcal{B}$ consists of functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ and natural transformations $\eta : \mathrm{id}_{\mathcal{A}} \to GF$ and $\epsilon : FG \to \mathrm{id}_{\mathcal{B}}$ such that $(G\epsilon B)(\eta GB) = \mathrm{id}_{GB}$ and $(\epsilon FA)(F\eta A) = \mathrm{id}_{FA}$; these equations are the so-called triangle identities. In this situation, one also says "F and G are adjoint functors," "F is the left adjoint" and "G is the right adjoint." The basic properties of functorial adjoint connections closely correspond to the basic properties of functorial Galois connections.

1. The categories $\mathrm{fix}(\eta)$ and $\mathrm{fix}(\epsilon)$ – containing all objects such that ηA (resp ϵB) is an isomorphism – are equivalent.

2. There is a natural bijection $\mathcal{A}(A, GB) \to \mathcal{B}(FA, B)$ given by

$$(f : A \to GB) \mapsto (\epsilon B)F(f) : FA \to B.$$

The inverse map is

$$(g : B \to FA) \mapsto G(f)(\eta A) : A \to GB.$$

3. F preserves colimits; G preserves limits. Borceux [6, Volume 1, 3.2.2]

4. Each ηA has the following universal property: if $f : A \to GB$, there is a unique map $\overline{f} = (\epsilon B)F(f) : FA \to B$ which makes the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "\eta A"] \arrow[dr, "f"'] & GFA \arrow[d, "G\overline{f}"] \\
 & GB
\end{tikzcd}$$

Each ϵB has the analogous universal property: if $f : FA \to B$, there is a unique $\overline{f} = G(f)(\eta A) : A \to GB$ such that

$$\begin{tikzcd}
B & FGB \arrow[l, "\epsilon B"] \\
 & FA \arrow[ul, "f"] \arrow[u, "\overline{f}"']
\end{tikzcd}$$

commutes. Any functor for which there is a natural transformation η with the above universal property is part of an adjoint connection.

For discussions of adjoint functors, see MacLane [21, Chapter IV], Herrlich and Strecker [13, Sections 26, 27, 28], or Borceux [6, Volume 1, Chapter 3].

There is a criterion for determining when functors are adjoints, which corresponds to Theorem 2.2.6.

Theorem 2.5.3. *(***Adjoint Functor Theorem***,* [21, V.6.2]*) Let $\mathcal{A}$ be a complete category. A functor $G : \mathcal{A} \to \mathcal{B}$ has a left adjoint if and only if*

1. *G preserves limits, and*

2. *(solution set condition) for each* $B \in \mathcal{B}$ *there is a set* I *and an* I*-indexed family of maps* $f_i : B \to GA_i$ *such that any map* $f : B \to GA$ *can be written as* $h = (Gt)f_i$ *for some* $i \in I$ *and* $t : A_i \to A$*.*

CHAPTER 3
ALGEBRAS OF A MONAD

Monads (a.k.a. triples, a.k.a. standard constructions) and their (Eilenberg-Moore) algebras provide a concise formulation of many important categorical aspects of universal algebra. The category of algebras for a monad has special properties, which are summarized in Section 3.1; a category of algebras is often complete and cocomplete, and always has "free objects." Each (functorial) adjoint connection induces a monad; the correspondence between adjunctions and monads is discussed in Section 3.2. In Section 3.3, the question of when an adjoint connection connects $\mathcal{A}$ to a category of algebras is addressed. In Section 3.4, the question "when does the composite of two monadic adjunctions yield a monadic adjunction?" is discussed.

The results in this chapter are reasonably well known. MacLane [21, Chapter VI], Borceux [6, Volume 2, Chapter 4], Barr and Wells [4, Chapters 3 and 9], Manes [23] and the introduction to the seminar notes [1] contain good expositions of monads from various perspectives.

3.1 Categories of Algebras

Definition 3.1.1. A *monad* $\mathbf{T} = (T, \eta, \mu)$ on $\mathcal{A}$ consists of a functor

$$T : \mathcal{A} \to \mathcal{A},$$

a natural transformation

$$\eta : \mathrm{id}_{\mathcal{A}} \to T,$$

and a natural transformation

$$\mu : T^2 \to T,$$

such that the following identities (expressed by commutative diagrams) hold: the unit laws –

$$\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\
& T &
\end{tikzcd}$$

and the associative law –

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd} .$$

(In these diagrams, T^n denotes the n-fold composite of T with itself.)

Intuitively, TA is the free object on A; ηA is the "insertion of variables" map; μA is the "semantic composition," i.e., a map which allows one to view a polynomial with polynomial variables as a polynomial. See Example 3.2.2 for a concrete illustration of the roles of T, η and μ.

Definition and Remarks 3.1.2. Let $\mathbf{T}$ be a monad on $\mathcal{A}$. A *T-algebra* (A, a) consists of $A \in \mathrm{Obj}(\mathcal{A})$ and $a : TA \to A$ (the so-called structure map) such that $a(\eta A) = \mathrm{id}_A$ (unit law) and

$$\begin{tikzcd}
T^2A \arrow[r, "Ta"] \arrow[d, "\mu"] & TA \arrow[d, "a"] \\
TA \arrow[r, "a"] & A
\end{tikzcd}$$

(associative law) commutes. A homomorphism $f : (A, a) \to (B, b)$ of T-algebras is an $\mathcal{A}$-map f such that

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "a"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"] & A
\end{tikzcd}$$

commutes. The category of all T-algebras and T-algebra homomorphisms is denoted $\mathcal{A}^T$.

There is a forgetful functor $U^T : \mathcal{A}^T \to \mathcal{A}$; it is defined by $U^T(A, a) = A$ and $U^T(f) = f$. U^T has a left adjoint $F^T : \mathcal{A} \to \mathcal{A}^T$ defined by

$$F^T(A) = (TA, \mu A)$$

and

$$F^T(f) = Tf.$$

The associated natural transformations are

$$\eta^T = \eta : \mathrm{id}_{\mathcal{A}} \to U^T F^T = T$$

and

$$\epsilon^T : F^T U^T \to \mathrm{id}_{\mathcal{A}^T} : \epsilon^T(A, a) := a.$$

[21, VI.2.Theorem 1]

Limits in $\mathcal{A}^T$ are "computed in $\mathcal{A}$" in the following sense.

Proposition 3.1.3. [4, 3.3.4], [6, Volume 2, 4.3.1] *Suppose (T, η, μ) is a monad on $\mathcal{A}$. If D is a diagram in $\mathcal{A}^T$ such that $U^T(D)$ has a limit $(L, p_{U(A)})$, then there is a unique structure map $\ell : TL \to L$ such that each $p_{U(A)}$ is a T-algebra homomorphism.*

Proof. Let D be a diagram in $\mathcal{A}^T$, such that $U^T D$ has a limit (L, ℓ_A). For each $A \in \mathrm{Obj}(D)$, name the structure map $s_A : TA \to A$. The goal is to produce a structure map $s : TL \to L$ such that each $\ell_A : L \to A$ is a T-algebra map.

The requirement that each ℓ_A is T-algebra map amounts to: for each A, the diagram below commutes.

$$\begin{tikzcd}
FL \arrow[r, "F\ell_A"] \arrow[d, "s"] & FA \arrow[d, "s_A"] \\
L \arrow[r, "\ell_A"] & A
\end{tikzcd}$$

Since (L, ℓ_A) is a source for UD, $(FL, s_A(F\ell_A))$ is a source for UD. Thus, there is a unique $\mathcal{A}$ map $s : FL \to L$ making each diagram above commute.

Checking the unit law, $s(iL) = \mathrm{id}_L$: the diagram

$$\begin{tikzcd}
A \arrow[rrr, "iA"] & & & FA \arrow[ddd, "s_A"] \\
& L \arrow[ul, "\ell_A"'] \arrow[r, "iL"] \arrow[dr, equal] & FL \arrow[ur, "F\ell_A"] \arrow[d, "s"] & \\
& & L \arrow[dr, "\ell_A"] & \\
& & & A
\end{tikzcd}$$

commutes using the definition of s, and that i is a natural transformation. Thus, for each A, $\ell_A = \ell_A s(iL)$. From the uniqueness of the map from the limit of D to any other source for D, it follows that $\mathrm{id}_L = s(iL)$.

A similar "comparision of squares diagram" can be used to verify that the algebra associative law holds. $\square$

Corollary 3.1.4. *If $\mathcal{A}$ is complete and $\mathbf{T}$ is a monad on $\mathcal{A}$, then $\mathcal{A}^T$ is complete.*

If $\mathcal{A}$ is cocomplete, $\mathcal{A}^T$ is often also cocomplete. The following theorem was originally proved with fewer hypotheses in Linton [20]. Other expositions are given in Borceux [6, Volume 2, 4.3.4] and Barr and Wells [4, Section 9.3].

Theorem 3.1.5. *Let $\mathcal{A}$ be cocomplete and $\mathbf{T}$ be a monad on T. $\mathcal{A}^T$ is cocomplete if and only if $\mathcal{A}^T$ has coequalizers.*

3.2 Adjoint Connections induce Monads

The following proposition gives a correspondence between monads and adjoint connections. It should be emphasized that the correspondence is *not* bijective. Each monad gives rise to a unique adjoint connection, but in general many adjunctions induce the same monad.

Proposition 3.2.1. Correspondence between monads and adjoint connections

1. *Let* $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$, $\eta : \mathrm{id}_{\mathcal{A}} \to GF$, $\epsilon : FG \to \mathrm{id}_{\mathcal{B}}$ *be an adjoint connection. Then* $(GF, \eta, G\epsilon F)$ *is a monad on* $\mathcal{A}$.

2. *If* $\mathbf{T} = (T, \eta, \mu)$ *is a monad, then*

$$G^T F^T = T,$$

$$\eta^T = \eta,$$

and

$$G^T \epsilon^T F^T = \mu.$$

The proof of the preceding Proposition consists of verifying identities: the triangle identities for adjoint connections imply the unit laws; the associative law holds because the square defining horizontal compositions commutes. Details are given in MacLane [21, VI.2.Theorem 1].

This correspondence allows construction of many examples of monads. Often, but not always, "a naturally occurring" adjoint connection corresponds to the category of algebras over the induced monads.

Example 3.2.2. Consider the category **Grp** of groups. The forgetful functor $U_{\mathbf{Grp}} : \mathbf{Grp} \to \mathbf{Set}$ and the free group functor $F_{\mathbf{Grp}} : \mathbf{Grp} \to \mathbf{Set}$ form an adjoint connection between **Grp** and **Set**. Recall that $F_{\mathbf{Grp}}(X)$ is the set of all reduced words

$$x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n},$$

where $x_i \in X$, $s_i \in \mathbb{Z}$; a word is *reduced* if for all i with $2 \leq i \leq n$, $x_{i-1} \neq x_i$. The operation on $F_{\mathbf{Grp}}(X)$ is concatenation of words, followed by reduction. $T = U_{\mathbf{Grp}}F_{\mathbf{Grp}}$ is the functor part of the induced monad; $\eta : \mathrm{id}_{\mathbf{Set}} \to T$ is the natural transformation whose component at X sends $x \in X$ to word x^1; if $w_1^{a_1} \cdots w_n^{a_n} \in T^2(X)$ and for each i, $w_i = x_{i,1}^{b_{i,1}} \cdots x_{i,m(i)}^{b_{i,m(i)}}$, then

$$(\mu X)(w_1^{a_1} \cdots w_n^{a_n}) = (x_{1,1}^{b_{1,1}} \cdots x_{1,m(1)}^{b_{1,m(1)}})^{a_1} \cdots (x_{n,1}^{b_{n,1}} \cdots x_{n,m(n)}^{b_{n,m(n)}})^{a_n}.$$

Each group G is an T-algebra; the group multiplication and inversion give a map from the free group on the underlying set of G to G. Conversely, each T-algebra structure gives a group multiplication and inversion. **Grp** is equivalent to the category of $\mathbf{Set}^T$. (This example is typical in the sense that any category of finitary algebras – in the sense of universal algebra – is also a category of monad algebras. The parts T, η, and μ of a monad generally have the same roles as in this example.)

Example 3.2.3. Some (non-trivial) adjunctions involving **Set** induce the trivial monad. For example, consider the category **Top** of topological spaces and continuous maps. The forgetful functor

$$U_{\mathbf{Top}} : \mathbf{Top} \to \mathbf{Set}$$

has a left adjoint

$$F_{\mathbf{Top}} : \mathbf{Set} \to \mathbf{Top},$$

which sends a set X to the discrete topological space with underlying set X. One easily checks that $T = G_{\mathbf{Top}}F_{\mathbf{Top}}$ is the identity functor on **Set**, and that all associated natural transformations are identities.

Definition 3.2.4. Suppose $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$, $\eta : \mathrm{id}_{\mathcal{A}} \to GF$, and $\epsilon : FG \to \mathrm{id}_{\mathcal{B}}$ forms an adjoint connection. By Proposition 3.2.1, the adjoint connection induces a monad $(T = GF, \eta, G\mu F)$. If $\mathcal{B}$ is equivalent to $\mathcal{A}^T$, then we say the adjoint connection (sometimes just the right adjoint G) is *monadic*. Often a concrete category $\mathcal{B}$ has a "canonical" forgetful functor $G : \mathcal{B} \to \mathcal{A}$; in this case, one may even say $\mathcal{B}$ is monadic over $\mathcal{A}$ – omitting mention of G.

These examples illustrate the qualitatively different behaviors of monadic and non-monadic adjoint connections. Monadic categories are determined by the combinatorial structure of the free algebra functor F^T. Most "relational" categories – like **Top**, $\mathbb{P}$, the category of graphs, etc. – have free functors, but these free functors do not add any structure to the underlying set; they merely attach the "most discrete" possible relation to the given set.

3.3 Detecting Categories of Algebras

Because of the special properties of $\mathcal{A}^T$, the question of when an adjoint connection is monadic has great practical importance.

Remark 3.3.1. Let (A, a) be a T-algebra. Because of the associative and unit laws, and because η is natural, the following equations hold:

$$\begin{aligned} &\text{(nat)} && (Ta)(\eta TA) = (\eta A)a, \\ &\text{(unit1)} && (\mu A)(\eta TA) = \mathrm{id}_{TA}, \\ &\text{(unit2)} && a(\eta A) = \mathrm{id}_A, \text{ and} \\ &\text{(assoc)} && a(Ta) = a(\mu A). \end{aligned}$$

These equations imply that $a = \mathrm{coeq}(\mu A, Ta)$: for if $b : TA \to B$ right-identifies μA and Ta, then $b(\eta A) : A \to B$ and

$$\begin{array}{rcll} b(\eta A)a & = & b(Ta)(\eta TA) & \text{by (nat)} \\ & = & b(\mu A)(\eta TA) & \text{since } b \text{ right identifies} \\ & = & b & \text{by (unit1).} \end{array}$$

One verifies that $f = b(\eta A)$ the unique map $f : A \to B$ with $fa = b$; for if $fa = b$, then $f = fa(\eta A) = b(\eta A)$.

These equations give a great deal of information about a. Thus, the key hypothesis in Beck's theorem – the criterion for when an adjunction is monadic – is the preservation and reflection coequalizers obeying the equations described above.

Using the above described equations for motivation we offer the following definitions.

Definition and Remarks 3.3.2. Consider maps $f, g : A \to B$.

1. If $e : B \to C$ has the feature that for any functor F, $Fe = coeq(Ff, Fg)$ then we say e is an *absolute coequalizer*.
2. If there are maps $e : B \to C$, $s_C : C \to B$ and $s_B : B \to A$ such that

   $$fs_B = s_c e,$$

   $$\mathrm{id}_A = gs_B,$$

   $$\mathrm{id}_C = es_C,$$

   $$ef = eg,$$

   then we say e is a *split coequalizer*.

Note that for every T-algebra (A, a), a is a split coequalizer of Ta and μA. It is also easy to see that every split coequalizer is absolute.

The following theorem is due to Beck (unpublished). Linton [19] contains a detailed discussion of variations on the theorem. The theorem was originally phrased in terms of split coequalizers only; Pare [26] refined the theorem to include the "absolute coequalizer" condition. Many variations on the hypotheses exist; the version stated here is found in MacLane [21, VI.7.1]. Other expositions of the theorem may be found in Barr and Wells [4, Section 3.3] and Borceux [6, Volume 2, Section 4.4].

Theorem 3.3.3. *Let* $(F : \mathcal{A} \to \mathcal{B}, G : \mathcal{B} \to \mathcal{A}, \eta, \epsilon)$ *be an adjoint connection, and* $T = GF$, $\eta = \eta$, *and* $\mu = G\epsilon F$ *be the associated monad. The following conditions are equivalent:*

1. *The adjunction* (F, G, η, ϵ) *is monadic.*
2. *If* $f, g : A \to B \in \mathrm{Map}(\mathcal{B})$ *and the pair* (Gf, Gg) *has an absolute coequalizer* e', *then* $e = \mathrm{coeq}(f, g)$ *exists and* $Ge = e'$.

3. *If* $f, g : A \to B \in \mathrm{Map}(\mathcal{B})$ *and the pair* (Gf, Gg) *has a split coequalizer* e'*, then* $e = \mathrm{coeq}(f, g)$ *exists and* $Ge = e'$.

Recognizing monadic adjunctions is complex because compositions of monadic functors are not generally monadic.

Example 3.3.4. The category **Ab** – of abelian groups – is monadic over **Set**, and **tfAb** (see Example 2.1.3) is monadic over **Ab**, because any reflection is monadic. Each free abelian group is torsion free, so the monad on **Set** induced by the free abelian group functor is the same as the monad induced by the "free torsion free abelian group functor." Thus, if **tfAb** were monadic over **set**, then **tfAb** and **Ab** would be equivalent categories – both equivalent to a suitable category of monad algebras. The categories **tfAb** and **Ab** are obviously not equivalent. (The preceding example is summarized from Borceux [6, Volume 2, Example 4.6.4].)

If $G : \mathcal{A} \to$ **Set**, the hypotheses of Beck's theorem can be reformulated to make it easier to check whether G is monadic.

Theorem 3.3.5. [11, Theorem 4.2] *Let* $G : \mathcal{A} \to$ **Set** *be a functor with a left adjoint. Suppose that* $\mathcal{A}$ *is complete and has coequalizers. The following are equivalent:*

1. G *is monadic.*
2. G *satisfies the following conditions*
   - G *preserves and reflects regular epimorphisms;*
   - *if* $f : GA \to X$ *is an isomorphism, then there is a unique map* $g : A \to B$ *such that* $Gg = f$*;*
   - G *reflects kernel pairs.*

The following lemma provides useful information concerning when a functor between two categories of algebras is monadic.

Lemma 3.3.6. [6, Volume 2, Corollary 4.5.7] *Suppose* $U : \mathcal{B} \to \mathcal{A}$*,* $V : \mathcal{C} \to \mathcal{A}$*, and* $Q : \mathcal{B} \to \mathcal{C}$ *are functors. If* $U = VQ$*,* U *and* V *are monadic, and* $\mathcal{B}$ *has coequalizers, then* Q *is monadic. In particular,* Q *has a left adjoint.*

3.4 Distributive Laws

Compositions of monadic adjunctions are not generally monadic; when a composition of monadic adjunctions is monadic, it indicates a distributive law between the two structures. The following section summarizes the results later needed from Beck's [5]; this exposition follows Beck's notation, except that composition of functions here will read right-to-left.

For the duration of the section, assume there are two monads $\mathbf{T} = (T, \eta^T, \mu^T)$ and $\mathbf{S} = (S, \eta^S, \mu^S)$ over some base category $\mathcal{A}$. For any monad, we use F and U, with superscripts for the name of the monad, to denote the free algebra functor and forgetful functor associated with a monad, respectively. See 3.1.2 for the definitions of F and U.

This material is rather abstract, so it helps to have an example in mind: after each definition and theorem, we will illustrate what it means using **S** – the free monoid monad over **Set** – and **T** – the free abelian group monad over **Set**; the composite monoid **TS** gives the free ring. For a set X, SX consists of all "strings" from X, with concatenation as the operation: S acts on functions by

$$Sf(x_1 x_2 \cdots x_n) = f(x_1)f(x_2)\cdots f(x_n).$$

TX consists of all formal (finite) linear combinations of elements of X, with integer coefficients. T acts on functions in the expected way.

Despite the concrete illustrations in terms of these monads, all theorems and defintions will apply to any monads **S** and **T**.

Definition and Remarks 3.4.1. A *distributive law* of **S** over **T** is a natural transformation

$$\ell : ST \to TS$$

satisfying the compatibility conditions:

$$\begin{tikzcd}
 & T \arrow[dl, "\eta^S T"'] \arrow[dr, "T\eta^S"] & \\
ST \arrow[rr, "\ell"] & & TS
\end{tikzcd}
\qquad
\begin{tikzcd}
 & S \arrow[dl, "S\eta^T"'] \arrow[dr, "\eta^T S"] & \\
ST \arrow[rr, "\ell"] & & TS
\end{tikzcd}$$

and

$$\begin{tikzcd}
SST \arrow[r, "S\ell"] \arrow[d, "\mu^S T"] & STS \arrow[r, "\ell S"] & TSS \arrow[d, "T\mu^S"] \\
ST \arrow[rr, "\ell"] & & TS \\
STT \arrow[r, "\ell T"] \arrow[u, "S\mu^T"] & TST \arrow[r, "T\ell"] & TTS \arrow[u, "\mu^T S"']
\end{tikzcd}$$

If there is a distributive law $\ell : ST \to TS$ of $\mathbf{S}$ over $\mathbf{T}$, then the *composite monad* is $\mathbf{TS} = (TS, \eta^{TS} := \eta^T\eta^S, \mu^{TS} := \mu^T\mu^S(T\ell S))$. In the definitions of η^{TS} and μ^{TS}, juxtoposition denotes horizontal composition of natural transformations. The verification that **TS** actually defines a monad is omitted; for more information see Beck [5].

In the case when "**S**=free monoid" and "**T**=free abelian group," the natural transformation $\ell : ST \to TS$ expresses a product-of-sums as a sum-of-products in the usual way:

$$\prod_{k\in K}\sum_{i\in I} a_{i,k} \mapsto \sum_{c}\prod_{k\in K} a_{c(k),k}$$

where c ranges over all choice functions $c : K \to I$. It is a somewhat enlightening exercise to check that ℓ – defined this way – is a natural transformation $ST \to TS$.

A distributive law may of T over S also be viewed as a "way of lifting T to a monad over S."

Definition 3.4.2. $\mathbf{T}$ has a lifting into $\mathcal{A}^{\mathbf{S}}$, if there is a monad $\overline{\mathbf{T}}$ such that $TU^S = U^S\overline{T}$, $U^S\eta^{\overline{T}} = \eta^T U^S$ and $U^S\mu^{\overline{T}} = \mu^T U^S$. To sketch the situation, $\overline{T}$ is a lifting of T onto $\mathcal{A}^S$ if

$$\begin{tikzcd}
\mathcal{A}^S \arrow[r, "\overline{T}"] \arrow[d, "U^S"] & \mathcal{A}^S \arrow[d, "U^S"] \\
\mathcal{A} \arrow[r, "T"] & \mathcal{A}
\end{tikzcd}$$

commutes.

Theorem 3.4.3. *There is a bijective correspondence between liftings of* $\mathbf{T}$ *to* $\mathcal{A}^S$ *and distributive laws of* $\mathbf{S}$ *over* $\mathbf{T}$.

The proof is outlined in Beck [5]. The correspondence is defined as follows: if ℓ is a distributive law, and (A, a) is an S-algebra, then $\overline{T}$ is defined as follows:

$$\overline{T}(A, a) = (TA, (Ta)(\ell A)),$$

$$\eta^{\overline{T}}(A, a) = \eta^T A : (A, a) \to \overline{T}(A, a),$$

and

$$\mu^{\overline{T}}(A, a) = \mu^T A : \overline{TT}(A, a) \to \overline{T}(A, a).$$

Consider the following diagram:

$$\begin{tikzcd}
 & STA \arrow[d, "\ell A"] \\
SA \arrow[ur, "S\eta^T A"] \arrow[r, "\eta^T SA"] \arrow[d, "a_S"] & TSA \arrow[d, "T(a_S)"] \\
A \arrow[r, "(\eta^T A)"] & TA
\end{tikzcd}$$

the square commutes because η^T is a natural transformation; the triangle commutes because ℓ is compatible with η^S. Since the perimeter of the diagram commutes, $\eta^T A$ is an S-algebra map. A similar diagram, which uses the compatibility between ℓ and μ^T, shows μ^T is an S-algebra map. Since the underlying maps are defined in $\mathcal{A}$ and obey the monad laws, $\eta^{\overline{T}}$ and $\mu^{\overline{T}}$ also obey the monad laws. If $\overline{\mathbf{T}}$ is a lifting of $\mathbf{T}$ over $\mathcal{A}^S$, ℓ is defined to be the following composition:

$$ST \xrightarrow{ST\eta^S} STS = STU^SF^S = U^SF^SU^S\overline{T}F^S \xrightarrow{U^S\epsilon^S\overline{T}F^S} U^S\overline{T}F^S = ST.$$

(In the above equation, ϵ^S is the counit of the adjunction $(F^S, U^S, \eta^S, \epsilon^S)$; $\epsilon^S(A, a) = a$.) After some detailed computation, one verfies that this is a distributive law, and that the correspondences described are mutually inverse.

Corollary 3.4.4. *If $\mathbf{T}$ has a lifting to $\mathcal{A}^S$, then there is a composite monad $\mathbf{TS}$.*

Theorem 3.4.5. *Suppose $\ell : ST \to TS$ is a distributive law. The categories $\mathcal{A}^{\mathbf{TS}}$ and $(\mathcal{A}^S)^{\overline{T}}$ are equivalent.*

Let us consider this in more detail. An object in $(\mathcal{A}^S)^{\overline{T}}$ consists of an $\mathcal{A}$-object, A, along with an S-structure $a_S : SA \to A$, T-structure $a_T : TA \to A$, and S-structure $t : STA \to A$ for TA such that the following diagram commutes.

$$\begin{tikzcd}
STA \arrow[r, "Sa_T"] \arrow[d, "t"] & SA \arrow[d, "a_S"] \\
TA \arrow[r, "a_T"] & A
\end{tikzcd}$$

Given a TS-algbra (A, a), A has a T-structure a_T map defined by

$$a_T = a(T\eta^S A) : TA \to A$$

and an S-structure map a_S defined by

$$a_S = a(\eta^T SA) : SA \to A.$$

The following diagram – which expresses the distributivity of the structure maps – commutes.

$$\begin{tikzcd}
STA \arrow[rr, "\ell A"] \arrow[d, "Sa_T"] & & TSA \arrow[d, "Ta_S"] \\
SA \arrow[dr, "a_S"] & & TA \arrow[dl, "a_T"] \\
& A &
\end{tikzcd}$$

Thus we have a map $\mathcal{A}^{TS} \to (\mathcal{A}^S)^{\overline{T}}$, given by,

$$(A, a) \in \mathcal{A}^{TS} \mapsto (A, a_T, a_S, Ta_S(\ell A))$$

which is functorial, because both η^T and η^S are natural.

The inverse functor $(\mathcal{A}^S)^T \to \mathcal{A}^{TS}$, maps (A, a_T, a_S, t) to $a_S S(a_T) = a_T t$. The verifications for these assertions is given in Beck [5].

CHAPTER 4
GENERATING SUBMONADS

In this chapter, a technique for creating monads is laid out; before proceeding formally, let us consider a rough outline of the technique. Suppose (T, η, μ) is a monad on a reasonable category, and that $f : F \to T$ is a subfunctor of T. The goal is to extend F to a submonad of T.

Intuitively speaking, F is a natural collection of polynomials. One should therefore require that for all A, $(\eta A)(A) \subseteq F(A)$ – i.e., that for each A and $a \in A$, FA contains the "constant polynomial with value a". F is not generally "closed under composition"; i.e., it is not generally true that $(\mu A)(F^2 A) \subseteq A$. To correct this problem, we start with F and iteratively add polynomials obtained by composing members of FA.

The purpose of the chapter is to formalize the preceding vague outline. The author abstracted and clarified [25, Proposition 3.4], which may be viewed as a special case of this result. Thus, the main results of this chapter are called Meseguer's Lemmas – 4.2.1, 4.2.2, 4.2.6, and 4.2.7 – to acknowledge the analogy which prompted the technique. The author believes the general formulation of the technique is new.

Section 4.1 details requirements on a "reasonable category", explains precisely what is meant by "subfunctor", and gives methods for constructing subfunctors. Section 4.2 gives a proof of Meseguer's Lemmas. Section 4.3 describes an example showing the necessity of a technical hypothesis of the lemmas.

4.1 Subfunctors

This section describes a general theory of subfunctors. A subfunctor of $F : \mathcal{X} \to \mathbf{Set}$ is a natural transformation $\eta : E \to F$ such that each component is a subset inclusion. The concept “subfunctor” is quite useful, but in some categories such as $\mathbb{P}$, **Top**, and **Locale** (explained in Example 4.1.2), there is either

- no obvious meaning for subset inclusion (in **Locale**), or
- more than one possible structure on each subset (in $\mathbb{P}$ and **Top**).

The following section expounds a theory in which “subfunctor” means “natural transformation whose components are all extremal mono”.

Axioms 4.1.1. Throughout, the base category $\mathcal{A}$ is assumed to have the following features:

1. $\mathcal{A}$ is complete.
2. $\mathcal{A}$ is an (epi, extremal mono) - category. Recall that an (epi, extremal mono) - category is a category with the features that
    - For each map f, it is possible to factor f as $f = me$ where e is epi and m is extremal mono. This factorization is unique, in the sense that if $f = m'e'$ is another way of writing f as an epi followed by an extremal mono, there is an isomorphism i such that

      $$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "e'"'] & \cdot \arrow[d, "m"] \arrow[dl, "i"] \\ \cdot \arrow[r, "m'"'] & \cdot \end{tikzcd}$$

      commutes.
    - if a and b are extremal mono and ab is defined, then ab is extremal mono.
3. $\mathcal{A}$ is extremally-wellpowered. This means: for each A, there is a set

   $$\{f_\lambda : S_\lambda \to A\}$$

   of extremal monomorphisms, such that if $m : T \to A$ is any extremal monomorphism, there is an index λ and isomorphism i such that $f_\lambda = mi$.

If $\mathcal{A}$ has these features, then it is an *SF-category.* ("SF" stands for "subfunctor".)

Example 4.1.1. By Herrlich and Strecker [13, 34.5], any wellpowered category with intersections and equalizers is an (epi, extremal mono) - category. So if $\mathcal{A}$ is complete and wellpowered then $\mathcal{A}$ is an SF-category.

It follows that $\mathbb{P}$, **Set**, **Top** and practically any reasonable category of topological spaces, is an SF-category.

Example 4.1.2. The category of frames – i.e., complete lattices in which

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$$

holds for all elements a and subsets S – is complete and cocomplete. This category also has (regular epi, mono) factorizations. Thus, **Locale**, the dual category to the category of frames, is an SF-category. **Locale** is extremally-wellpowered, but not wellpowered. The author's interest in locales motivated him to use the given definition (which only requires extremal-wellpoweredness) for SF-category rather than defining SF-category to mean "complete and wellpowered", so that the theory of subfunctors would apply to localic subfunctors in addition to the previous examples. (For background on frames and their relation to point-set topology, see Isbell [15], Johnstone [16], Joyal and Tierney[17] and Madden [22].)

Lemma 4.1.3. *Assume that $\mathcal{A}$ is an (epi, extremal mono) - category. Then*

1. **(diagonalization)** [13, 33.3] *If $ge = mf$, where e is epi and m is extremal mono, then there exists k such that $mk = g$ and $ke = f$.*
2. [13, 34.2(2)] *Any intersection of extremal subobjects is extremal.*
3. [13, 34.2(3)] *Pullbacks of extremal monomorphisms are extremal mono.*

Definition and Remarks 4.1.4. Defining $\mathrm{Sub}(A)$ – **the lattice of extremal subobjects:** In an SF-category $\mathcal{A}$, the lattice of (equivalence classes) of extremal subobjects has particularly nice structural features.

An *extremal subobject* of A is an extremal monomorphism $s : S \to A$. Define a preorder on the class of extremal subobjects by

$$(s : S \to A) \subseteq (t : T \to A) \iff \exists c : S \to T\,,\ s = tc.$$

For brevity, one often only mentions the map or object part of an extremal subobject. To avoid confusion, the same letter will be used to denote both parts, with the lower case letter used for the function.

Since t is mono, there is at most one map c which manifests $s \subseteq t$. If $s \subseteq t$ and $t \subseteq s$, then there are c_1 and c_2 such that $s = tc_1$ and $t = sc_2$. So $t = tc_1c_2$ and $s = sc_2c_1$. Since c_2c_1 shows $s \subseteq s$, $c_2c_1 = \mathrm{id}_S$. Similarly, $c_1c_2 = \mathrm{id}_T$. One identifies extremal subobjects s and t if $s \subseteq t$ and $t \subseteq s$, or equivalently, when there is an isomorphism c such that $s = tc$. The set of equivalance classes under this relation is denoted $\mathrm{Sub}(A)$.

Any map c, which exhibits $s \subseteq t$, is extremal mono. For if $ca = cb$, then $sa = tca = tcb = sb$, so (since s is mono) $a = b$; so c is mono. If $c = me$, where e is epi, then $s = tc = tme$. Since s is extremal mono, e must be an isomorphism. Thus c is extremal mono.

Recall that the *intersection* of a set M of monomorphisms with a common codomain is the limit of the diagram generated by M. Since $\mathcal{A}$ is complete and intersections of extremal monomorphisms are extremal, $\mathrm{Sub}(A)$ is a complete lattice, with meet operation – $\bigcap$ – and join operation – $\bigcup$. In general, joins in $\mathrm{Sub}(A)$ are not disjoint unions; in general, $\bigcup A_i$ is computed using

$$\bigcup A_i = \bigcap \{A' : \forall i,\ A_i \subseteq A'\}.$$

(See Borceux [6, Volume 1, 4.2.2, 4.2.3, 4.2.4].)

A map $f : A \to B$ induces an adjoint connection between $\mathrm{Sub}(A)$ and $\mathrm{Sub}(B)$. (See Borceux [6, Volume 1, 4.4.6]; Borceux's results are phrased in terms of "strong monomorphisms". Under our assumptions a map is strong mono if and only if it is extremal mono.) If $s : S \to A \in \mathrm{Sub}(A)$, use the (epi, extremal mono) - factorization to obtain a unique

$s' \in \mathrm{Sub}(B)$ such that

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
S \arrow[u, "s"] \arrow[r] & f^{+1}(S) \arrow[u, "s'"']
\end{tikzcd}$$

commutes. The notation $f^{+1}(S)$ is used for the image of S, to remind the reader of the analogy to ordinary set theoretic images of subsets under maps. The map $(s : S \to A) \mapsto (s' : S' = f^{+1}(S) \to B)$ is the left adjoint. The right adjoint $f^{-1} : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ is obtained by taking the pullback of $s : S \to B$ along f. Thus,

$$f^{+1}(S) \subseteq T \iff S \subseteq f^{-1}(T),$$

$$f^{+1}(\bigcup S_i) = \bigcup f^{+1}(S_i),$$

and

$$f^{-1}(\bigcap S_i) = \bigcap f^{-1}(S_i).$$

Definition and Remarks 4.1.5. Let $F : \mathcal{X} \to \mathcal{A}$ be a functor. A *subfunctor* E of F is a rule that selects an extremal subobject $eX : EX \to FX$ for each $X \in \mathrm{Obj}(\mathcal{X})$ such that if $f : X \to Y$, then

$$(Ff)^{+1}(EX) \subseteq EY.$$

Any such assignment E of F gives rise to a functor $E : \mathcal{X} \to \mathcal{A}$. $Ef : EX \to EY$ is defined to be the composition $EX \to (Ff)^{+1}(EX) \subseteq EY$.

Since $eY : EY \to FY$ is mono, Ef is uniquely determined by the condition that

$$\begin{tikzcd}
FX \arrow[r, "Ff"] & FY \\
EX \arrow[u, "eX"] \arrow[r, "Ef"] & EY \arrow[u, "eY"']
\end{tikzcd}$$

commutes. Thus, $(Ff)^{+1}(EX) \subseteq EY$ implies that Ef can be defined to make the above square commute.

The converse also holds: if there is a map Ef such that the square commutes, then $(Ff)^{+1}(EX) \subseteq EY$. To prove this, suppose there is a map Ef which makes the square commute. Use the unique factorization $Ef = ab$ where $a : A \to EY$ is extremal mono and b is epi. Note that $(Ff)(eX) = (eY)(Ef) = (eY)ab$ gives a factorization of $(Ff)(eX)$ into an epi b followed by an extremal mono $(eY)a$. Thus,

$$[(eY)a : A \to FT] = [(Ff)^{+1}(EX) \to FT]$$

and $(Ff)^{+1}(EX) \subseteq EY$. Thus, a subfunctor is exactly "a natural transformation whose components are all extremal monomorphisms".

Let Subfun(F) denote the class of subfuctors of $F : \mathcal{X} \to \mathcal{A}$. Define a preorder on Subfun(F) by

$$E_1 \subseteq E_2 \iff \forall A \in \mathrm{Obj}(\mathcal{A})\,,\, E_1(A) \subseteq E_2(A).$$

If $E_1 \subseteq E_2$, then, for each A, there is a unique extremal monomorphism cA such that $e_1A = (e_2A)(cA)$. One easily verifies that cA are the components of a natural transformation $c : E_1 \to E_2$; in fact, $c : E_1 \to E_2$ is a subfunctor.

The following constructions show that the class Subfun(F) behaves very much like Sub(A).

Construction 4.1.6. Subfun(F) *is complete. If* $\{E_i : i \in I\}$ *is any class of subfunctors, there is a supremum* $\bigcup E_i$ *and infimum* $\bigcap E_i$. *The supremum is given by* $(\bigcup E_i)(A) = \bigcup E_i(A)$. *The infimum is given by* $(\bigcap E_i)(A) = \bigcap E_i(A)$.

Proof. Since $\mathcal{A}$ is extremally wellpowered, for each A, the class $\{E_i(A) : i \in I\}$ has a representative set. Thus, the object-by-object definitions for $\bigcup E_i$ and $\bigcap E_i$ make sense. It is obvious that if $\bigcup E_i$ and $\bigcap E_i$ are subfunctors that they are the optimum bounds in the subfunctor lattice.

Let $f : X \to Y$ be any map. $A \mapsto \bigcap_i E_i(A)$ is a subfunctor because for each $i \in I$,

$$(Ff)^{+1}(\bigcap_i E_i(A)) \subseteq (Ff)^{+1}(E_i(A)) \subseteq E_i(B),$$

so $(Ff)^{+1}(\bigcap_i E_i(A)) \subseteq \bigcap_i E_i(B)$.

$A \mapsto \bigcup_i E_i(A)$ is a subfunctor because,

$$(Ff)^{+1}(\bigcup E_i(A)) = \bigcup_i (Ff)^{+1}(E_i(A)) \subseteq \bigcup_i E_i(B).$$

□

Construction 4.1.7. *Suppose $\alpha : F \to G$ is a natural transformation. Then there is an (order theoretic) adjoint conection*

$$\alpha^{+1} : \mathrm{Subfun}(F) \leftrightarrow \mathrm{Subfun}(G) : \alpha^{-1}.$$

The left adjoint α^{+1} is defined by $(\alpha^{+1}E)(A) = (\alpha A)^{+1}(EA)$. The right adjoint α^{-1} is defined by $(\alpha^{-1}E)(A) = \alpha^{-1}(EA)$.

Proof. The order on Subfun(F) is defined "object-by-object", and whenever

$$f : A \to B,$$

f^{+1} and f^{-1} form an adjoint connection between Sub(A) and Sub(B). So it suffices to check that $\alpha^{+1}E$ and $\alpha^{-1}E$ define subfunctors.

Let $E \in \mathrm{Subfun}(F)$, $f : X \to Y$ be a map, and consider the following diagram.

$$\begin{tikzcd}
(\alpha A)^{+1}(EA) \arrow[drr, "e'A"] & & & & & & (\alpha B)^{+1}(EB) \arrow[dll, "e'B"'] \\
& & GA \arrow[rr, "Gf"] & & GB & & \\
& (iii) & & (iv) & & (i) & \\
& & FA \arrow[uu, "\alpha A"] \arrow[rr, "Ff"] & & FB \arrow[uu, "\alpha B"] & & \\
& & (ii) & & & & \\
EA \arrow[uuuuu, "zA"] \arrow[uurr, "eA"] \arrow[rrrrrr, "Ef"] & & & & & & EB \arrow[uuuuu, "zB"'] \arrow[uull, "eB"']
\end{tikzcd}$$

The trapezoids (i) and (iii) are obtained by factoring $(\alpha B)(eB)$ and $(\alpha A)(eA)$, respectively; in each case z is the epi part and e' is the extremal mono part. The square (iv) expresses the naturality of α. The trapezoid (ii) expresses that $e : E \to F$ is a subfunctor.

To show $\alpha^{+1}E$ is a subfunctor of G, it suffices to show there is a map

$$k : (\alpha A)^{+1}(EA) \to (\alpha B)^{+1}(EB)$$

that makes the top trapezoid commute. For this, use the diagonalization property from Lemma 4.1.3. Define $f = zB(Ef)$, $g = (Gf)e'A$, $e = zA$ and $m = e'B$; note that $ge = mf$ with e epi and m extremal mono. Thus, the diagonalization property guarantees the desired k exists.

Assume $f : A \to B$. A similar diagram is used to verify that $\alpha^{-1}E$ is a subfunctor of F whenever $E \in \mathrm{Subfun}(G)$. The missing map $(\alpha A)^{-1}(EA) \to (\alpha B)^{-1}(EB)$ is obtained from the pullback property of $(\alpha B)^{-1}(EB)$. $\square$

Corollary 4.1.8. *Let* $\alpha : F \to G$ *be a natural transformation, and* $e : E \to F$ *be a subfunctor of* F*. Let* $e' : \alpha^{+1}E \to F$ *be the subfunctor described in Construction 4.1.7. There is a natural transformation* $z : \alpha^{+1}E \to F$ *such that for each object* A*,* $(\alpha A) = (e'A)(zA)$ *is the (epi, extremal mono)-factorization of* αA*.*

Proof. One defines e' and z as in the preceding proof. Examining the comparison of squares diagram used to produce $\alpha^{+1}Ef$, shows that z and e' are natural. ☐

Construction 4.1.9. *Assume* E, F, G, H *are functors* $\mathcal{A} \to \mathcal{A}$*. Suppose that*

1. $\alpha : E \to F$ *and* $\beta : G \to H$ *are subfunctors,*
2. *Either* E *or* F *preserves extremal monomorphisms.*

then the horizontal composition $\beta\alpha : EG \to FH$ *is a subfunctor.*

Proof. The horizontal composition of natural transformations is always natural. Hypothesis 2 implies that, for any A, $\beta\alpha A$ is an extremal monomorphism, because the class of extremal monomorphisms is closed under composition and

$$(\beta\alpha)A := (\alpha HA)(E\beta A) = (F\beta A)(\alpha GA),$$

by definition of horizontal composition (see Construction 2.3.5). ☐

4.2 Meseguer's Lemmas

Lemma 4.2.1. *Suppose* $\mathcal{A}$ *is an SF-category,* (T, μ, η) *is a monad on* $\mathcal{A}$*, and* F *is a subfunctor of* T*, such that* $(\eta A)^{+1}(A) \subseteq FA$*. There is a smallest subfunctor* $\overline{F}$ *of* T *such that* $F \subseteq \overline{F} \subseteq T$ *and for all* $A \in \mathrm{Obj}(\mathcal{A})$

$$(\mu A)(\overline{F}^2 A) \subseteq \overline{F}A.$$

If the equation

$$(\mu A)(G^2 A) \subseteq GA$$

holds for G, we say G *is closed under* μ.

Proof. Let $\mathfrak{F}$ denote the class of all subfunctors of T, that are larger than F and closed under μ. $\mathfrak{F}$ is nonempty because it contains T. By Construction 4.1.6, $\overline{F} = \bigcap \mathfrak{F}$ exists. Since $S \mapsto (\mu A)(S)$ is an order preserving map $\mathrm{Sub}(T^2A) \to \mathrm{Sub}(TA)$, so for each A $(\mu A)(\overline{F}^2 A) \subseteq \overline{F}A$. □

The preceding gives an easy candidate for the functor part of the monad generated by F. One needs a more complex argument if one wants detailed information about the natural transformations – related to η and μ – which make $\overline{F}$ into a monad.

The proofs of the following lemmas require detailed computation. For clarity, the goal of each paragraph in the proof is written in bold-face.

Lemma 4.2.2. *(Assume notation and hypotheses from Lemma 4.2.1.) There are natural transformations* $\overline{n} : \mathrm{id}_{\mathcal{A}} \to \overline{F}$ *and* $\overline{m} : \overline{F}^2 \to \overline{F}$ *which make* $(\overline{F}, n, m)$ *a monad.*

Proof. One defines four sequences of natural transformations:

1. $f_\lambda : F_\lambda \to T$ – the subfunctor generated at stage λ,
2. $n_\lambda : \mathrm{id}_{\mathcal{A}} \to F_\lambda$ – a natural transformation obtained from η by suitably modifying the domain and codomain,
3. $m_\lambda : F_\lambda^2 \to F_{\lambda+1}$ – a natural transformation obtained from μ by suitably modifying domain and codomain, and
4. $c_\lambda : F_\lambda \to F_{\lambda+1}$ – the inclusion.

Define $F_0 = F$; use the notation $f_0 : F_0 \to T$. m_0 and c_0 are defined according to the same pattern that defines later "m"s and "c"s, which is described below.

Defining $f_\lambda : F_\lambda \to T$ ($\lambda > 0$), $m_\lambda : F_{\lambda^2} \to F_{\lambda+1}$, and $c_\lambda : F_\lambda \to F_{\lambda+1}$. Assume that f_λ has been previously defined. Consider the following diagram:

$$\begin{tikzcd}
F^2_\lambda A \arrow[r, "f^2_\lambda A"] \arrow[d] & T^2A \arrow[d, "\mu A"] \\
(\mu A)^{+1}(F^2_\lambda) \arrow[r] \arrow[d] & TA \\
(\mu A)^{+1}(F^2_\lambda) \cup F_\lambda A \arrow[ur, "f_{\lambda+1}"] &
\end{tikzcd}$$

Given f_λ and μA, the square is obtained by (epi, extremal mono)-factorization of the composite $(\mu A)(f^2_\lambda A)$. All maps in the triangle are extremal monomorphisms, obtained by comparing subobjects of TA. Define

$$F_{\lambda+1}A = (\mu A)^{+1}(F^2_\lambda A) \cup F_\lambda A$$

and

$$m_\lambda : F^2_\lambda A \to F_{\lambda+1}A$$

to be the map shown on the left side of the diagram. It follows from Constructions 4.1.7 and 4.1.6 that $f_{\lambda+1} : F_{\lambda+1} \to T$ is a subfunctor. Note that

$$(f_{\lambda+1})(m_\lambda A) = (\mu A)(f^2_\lambda A)$$

and m_λ is natural. For bookkeeping purposes, let us call

$$c_\lambda : F_\lambda \to F_{\lambda+1}$$

the subfunctor which exhibits $F_\lambda \subseteq F_{\lambda+1}$.

If κ is a limit ordinal, define

$$F_\kappa = \bigcup\{F_\lambda : \lambda < \kappa\};$$

$f_\kappa : F_\kappa \to T$ is a subfunctor by 4.1.6.

Defining the sequence of "n_λ"s: The following definition of n_λ is not recursive; n_λ can be defined once we know f_λ, but the definition of f_λ does not involve n_λ at all. Consider the following commutative diagram of functors and natural transformations.

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{A}} \arrow[r, "z"] & (\eta \mathrm{id}_{\mathcal{A}})^{+1} \arrow[r, "e"] \arrow[d, "i_\lambda"] & T \\
& F_\lambda \arrow[ur, "f_\lambda"] &
\end{tikzcd}$$

The natural transformations z and e are defined by condition that $\eta = ez$ is the (epi, extremal mono)-factorization of η, as described in 4.1.8. By assumption and the construction of the sequence (F_λ),

$$(\eta \mathrm{id}_A)^{+1} \subseteq F_0 \subseteq F_\lambda.$$

Let i_λ denote the natural transformation such that $e = f_\lambda i_\lambda$; i_λ exists because $e \subseteq i$. Define $n_\lambda := i_\lambda z$. Evidently,

$$\eta = f_\lambda n_\lambda$$

and

$$n_{\lambda+1} = c_\lambda n_\lambda.$$

What happens when the sequence terminates: Since $\mathcal{A}$ is extremally wellpowered, for each A, the sequence $(F_\lambda A)_\lambda$ of subobjects of TA eventually terminates, say when $\lambda = \kappa_1$. Again using extremal wellpoweredness there is an ordinal, say κ_2, such that the sequence $(F_\lambda F_\kappa(A))_\lambda$ of subobjects of $TF_{\kappa_1}A$ terminates at κ_2. Define $\overline{n} = n_{\kappa_2}$, $\overline{f} = f_{\kappa_2}$ and $\overline{m} = m_{\kappa_2}$. These assignments give natural transformations; to check this, one considers a map $f : A \to B$, and chooses κ large enough that the subobject sequences (described above) terminate for both A and B. It should be clear that each subobject sequence $(F_\lambda A)_\lambda$ terminates at $\overline{F}A$.

Verifying that $(\overline{F}, \overline{n}, \overline{m})$ is a monad. To prove the unit laws, it suffices that for all λ,

$$(m_\lambda A)(F_\lambda n_\lambda A) = (m_\lambda A)(n_\lambda F_\lambda A) = c_\lambda A$$

(because, once the sequence terminates, the "$c_\lambda A$"s become identity maps on $F_\lambda A$). Fix any ordinal λ. Naturality of f_λ and n_λ implies

$$(Tn_\lambda A)(f_\lambda A) = (f_\lambda F_\lambda A)(F_\lambda n_\lambda A)$$

and

$$(n_\lambda TA)(f_\lambda A) = (F_\lambda f_\lambda A)(n_\lambda F_\lambda A);$$

Note that $\eta = f_\lambda n_\lambda$ and, by definition of horizontal composition,

$$f_\lambda^2 A = (Tf_\lambda A)(f_\lambda F_\lambda A) = (f_\lambda TA)(F_\lambda f_\lambda A).$$

Calculating, one finds

$$\begin{array}{rcll} (f_\lambda^2 A)(F_\lambda n_\lambda A) & = & (f_\lambda TA)(F_\lambda f_\lambda A)(F_\lambda n_\lambda A) & \text{def. noted above} \\ & = & (f_\lambda TA)(F_\lambda \eta A) & \text{noted above} \\ & = & (T\eta A)(f_\lambda A). & f_\lambda \text{ is natural} \end{array}$$

Thus,

$$(f_\lambda^2 A)(F_\lambda n_\lambda A) = (T\eta A)(f_\lambda A)$$

and (by a similar computation)

$$(f_\lambda^2 A)(n_\lambda F_\lambda A) = (\eta TA)(f_\lambda A).$$

To show $(m_\lambda A)(F_\lambda n_\lambda A) = c_\lambda A$, consider the following diagram:

$$\begin{tikzcd}
F_\lambda A \arrow[r, "F_\lambda(n_\lambda A)"] \arrow[d, "f_\lambda A"] & F_\lambda^2 A \arrow[d, "f_\lambda^2 A"] \arrow[dr, equal] & \\
TA \arrow[r, "T\eta A"] \arrow[dr, equal] & T^2 A \arrow[d, "\mu A"] & F_\lambda^2 A \arrow[l, "f_\lambda^2"] \arrow[d, "m_\lambda"] \\
& TA & F_{\lambda+1} A \arrow[l, "f_\lambda"]
\end{tikzcd}$$

the upper square commutes by an identity proved above; the lower square is the definition of m_λ; the upper triangle is trivial; the lower triangle is the unit law for (T, η, μ). Reading the perimeter of the diagram, one finds

$$f_\lambda A = (f_{\lambda+1}A)(m_\lambda A)(F_\lambda n_\lambda A).$$

By uniqueness of maps manifesting inequalities between subobjects,

$$c_\lambda A = (m_\lambda A)(F_\lambda n_\lambda A).$$

The diagram needed to prove

$$(m_\lambda A)(n_\lambda F_\lambda A) = c_\lambda A$$

is similar and omitted. This establishes the unit laws.

To prove the associative law for $(\overline{F}, \overline{n}, \overline{m})$, one may choose sufficently large λ, then draw a diagram comparing the associative squares for T and $\overline{F}$, using appropriate powers of f_λ to compare the corners. The comparison squares commute because of the definitions of m_λ and horizontal composition. $\square$

Definition 4.2.3. Assume that $\mathcal{A}$ is cocomplete. In $\mathcal{A}$, *unions of chains are colimits*, if whenever D is a diagram, where $\mathrm{Obj}(D)$ consists of a chain of extremal subobjects of A and $\mathrm{Map}(D)$ consists of all inclusions (in $\mathrm{Sub}(A)$) that exist among elements of $\mathrm{Obj}(D)$, the map $c : \mathrm{colim} D \to A$ induced by colimit properties is an extremal monomorphism.

In this case, $c : \mathrm{colimD} \to A$ is an extremal subobject which contains each element of $\mathrm{Obj}(D)$; thus there is an extremal monomorphism $\mathrm{colim} D \to \bigcup \mathrm{Obj}(D)$. Since $\bigcup \mathrm{Obj}(D)$ is the supremum of $\mathrm{Obj}(D)$ in the $\mathrm{Sub}(A)$, this map must be an isomorphism.

Remark 4.2.4. The condition that unions of chains are colimits is satisfied in many naturally occuring categories. For example, in “relational” categories like **Top**, $\mathbb{P}$, and **Set**, extremal subobjects are just subsets with the induced structure, so all unions are colimits.

Assume $\mathcal{A}$ is a category of finitary algebras. The (set theoretic)-union of a chain (A_i) of subobjects is a subobject. The (set theoretic)-union A' of (A_i) lies inside a smallest extremal subobject, which by our notation is $\bigcup A_i$. Evidently, the coproduct of the chain (A_i) of subobjects is A'. So the issue about whether "unions of chains are coproducts" amounts to checking whether $A' = \bigcup A_i$. This depends on the delicate issue of whether epimorphisms are surjective. If epis are not necessarily surjective, then $\bigcup A_i$ will be the largest subobject in which A' is epi, which will generally be larger than A'.

Definition 4.2.5. As above, assume $\mathcal{A}$ is an SF-category, (T, η, μ) is a monad, and F is a subfunctor of T which contains $\eta^{+1}\mathrm{id}_{\mathcal{A}}$. A *partial algebra* (A, a) is an object A equipped with a map $a : FA \to A$ such that $a(n_0 A) = \mathrm{id}_A$.

Let $\mathrm{pAlg}(F_0, n_0)$ denote the category of all partial algebras, with maps $f : (A, a) \to (B, b)$ such that $f : A \to B$ is an $\mathcal{A}$-map, and $fa = b(F_0 f)$.

Lemma 4.2.6. *Continue with situation and hypotheses from 4.2.2. Suppose $\mathcal{A}$ is cocomplete and unions of chains are colimits. Suppose (A, a) is a partial algebra. The partial algebra structure a extends to an $\overline{F}$-algebra structure map $\overline{a} : \overline{F}A \to A$ if and only if for each ordinal λ there is a unique $a_{\lambda+1}$ making the diagram below commute.*

$$\begin{tikzcd}
F_\lambda^2(A) \arrow[r, "F_\lambda a_\lambda"] \arrow[d, "m_\lambda"] & F_\lambda(A) \arrow[d, "a_\lambda"] \\
F_\lambda(A) \arrow[r, "a_{\lambda+1}"] & A
\end{tikzcd}$$

Proof. To prove a map $\overline{a} : \overline{F}A \to A$ exists, use transfinite induction. The hypothesis gives a_λ at sucessor ordinals. To construct a_λ, when λ is a limit ordinal use the coproduct property of $F_\lambda A = \bigcup_{\zeta<\lambda} F_\zeta(A)$.

Inductively one shows that

(ext) for each λ, $a_\lambda = a_{\lambda+1}(c_\lambda A)$ (each $a_{\lambda+1}$ extends the preceding a_λ),

(unit) for each λ, $a_\lambda(n_\lambda A) = \mathrm{id}_A$,

If (unit) holds for sucessor ordinals, then it holds for all ordinals. Suppose κ is a limit ordinal. The map a_κ is epi, because it is a limit of epimorphisms. Note that a_κ and $a_\kappa n_\kappa a_\kappa$ are both compatible maps $F_\kappa A \to A$; the definition of colimits implies that

$$a_\kappa = a_\kappa n_\kappa a_\kappa;$$

because a_κ is epi, $\mathrm{id}_A = a_\kappa n_\kappa$.

The algebra associative law holds for $(A, \overline{a})$. If $\overline{a} = a_\lambda$, then $a_\lambda = a_{\lambda+1}$, so by the hypothesis regarding the existence of a_λ such that $a_{\lambda+1}(m_\lambda A) = a_\lambda(F_\lambda a_\lambda)$ proves the associative law.

If (unit) holds for λ, then (ext) holds for λ. Note that the equation $a_{\lambda+1}(m_\lambda A) = a_\lambda(F_\lambda a_\lambda)$ defines $a_{\lambda+1}$. By induction hypothesis (unit),

$$(F_\lambda a_\lambda)(F_\lambda n_\lambda A) = \mathrm{id}_{F_\lambda A};$$

therefore precomposing both sides of the defining equation for $a_{\lambda+1}$ with $(F_\lambda n_\lambda A)$ shows

$$a_\lambda = a_{\lambda+1}(m_\lambda A)(F_\lambda n_\lambda A) = a_{\lambda+1}(c_\lambda A).$$

The second equality follows from $(m_\lambda A)(F_\lambda n_\lambda A) = (c_\lambda A)$, which was proved in verifying the unit laws for $\overline{\mathbf{F}}$.

If (unit) holds at λ, then it holds for $\lambda + 1$. Consider the following diagram.

$$\begin{tikzcd}
F_\lambda A \arrow[dr, "F_\lambda n_\lambda A"] \arrow[dd, equal] & & & A \arrow[lll, "n_\lambda A"'] \arrow[dl, "n_{\lambda+1}A"] \\
& F_\lambda^2 A \arrow[r, "m_\lambda A"] \arrow[dl, "F_\lambda a_\lambda"] & F_{\lambda+1}A \arrow[dr, "a_{\lambda+1}"] & \\
F_\lambda A \arrow[rrr, "a_\lambda"] & & & A
\end{tikzcd}$$

The left triangle commutes because (unit) holds for λ; the bottom trapezoid commutes by definition of $a_{\lambda+1}$; the top trapezoid commutes because

$$(n_\lambda A)(F_\lambda n_\lambda A)(m_\lambda A) = (n_\lambda A)(c_\lambda A) = n_{\lambda+1} A.$$

The first equality holds by proof of the unit law for $\overline{\mathbf{F}}$; the second equality holds because of the compatibility between "n_λ"s. Thus,

$$\begin{aligned} a_{\lambda+1}(n_{\lambda+1}) &= a_{\lambda+1}(m_\lambda A)(F_\lambda n_\lambda A)(n_\lambda A) \\ &= a_\lambda(F_\lambda a_\lambda)(F_\lambda n_\lambda A)(n_\lambda A) \\ &= a_\lambda(n_\lambda A) \\ &= \mathrm{id}_A; \end{aligned}$$

this establishes the unit law for $(A, \overline{a})$. □

Use the notation $\mathrm{Alg}(F_0, n_0)$ to denote the full subcategory of $\mathrm{pAlg}(F_0, n_0)$ containing objects satisfying the hypotheses of Lemma 4.2.6.

Lemma 4.2.7. *Suppose the preceding lemmas apply. Also suppose each* m_λ *is epi. The map* $(A, a) \mapsto (A, \overline{a})$ *induces an equivalence of categories; the functors involved are*

$$E : \mathrm{Alg}(F_0, n_0) \to \mathcal{A}^{\overline{F}},$$

defined by extension of structure and

$$R : \mathcal{A}^{\overline{F}} \to \mathrm{Alg}(F_0, n_0)$$

given by restriction.

Proof. E **is legitmately defined.** Lemma 4.2.6 defines E on objects. Maps in $\mathcal{A}^{\overline{F}}$ are $\mathcal{A}$-maps compatible with the structure. So it suffices to show that for any $\mathrm{Alg}(F_0, n_0)$ map $\phi : A \to B$,

$$\begin{tikzcd} F_\lambda A \arrow[r, "F_\lambda \phi"] \arrow[d, "a_\lambda"] & F_\lambda B \arrow[d, "b_\lambda"] \\ A \arrow[r, "\phi"] & B \end{tikzcd}$$

commutes for each ordinal λ. By definition of $\mathrm{Alg}(F_0, n_0)$, the square commutes for $\lambda = 0$. Suppose it commutes for λ; note that

$$\begin{array}{rcll} \phi a_{\lambda+1}(m_\lambda A) & = & \phi a_\lambda(F_\lambda a_\lambda) & \text{def. } a_{\lambda+1} \\ & = & b_\lambda(F_\lambda \phi)(F_\lambda a_\lambda) & \text{ind. hyp.} \\ & = & b_\lambda(F_\lambda b_\lambda)(F_\lambda^2 \phi) & \text{ind. hyp.} \\ & = & b_{\lambda+1}(m_\lambda B)(F_\lambda^2 \phi) & \text{def. } b_{\lambda+1} \\ & = & b_{\lambda+1}(F_{\lambda+1}\phi)(m_\lambda A) & \text{naturality } m_\lambda. \end{array}$$

Because m_λ is epi, one concludes $\phi a_{\lambda+1} = b_{\lambda+1}(F_{\lambda+1}\phi)$. At limit ordinals, properties of colimits insure that the diagram commutes. By transfinite induction, and the defintion of the extended structure, any map which preserves (F_0, n_0)-structure preserves $\overline{F}$-structure.

One also notes that restriction respects maps, because F_0 is a subfunctor of $\overline{F}$.

E and R form an equivalence. Obviously,

$$\forall (A, a_0) \in \mathrm{Obj}(\mathrm{Alg}(F_0, n_0)),\ RE(A, a_0) = (A, a_0).$$

The construction of the extended structure shows the restriction of any $\overline{F}$-structure to F_0 uniquely determines the $\overline{F}$-structure; hence

$$ER(A, a) = (A, a).$$

□

4.3 A Partial Algebra Which Does Not Extend

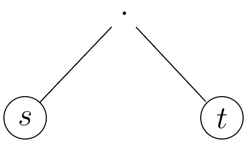
The category **Set** is an SF-category and unions are colimits, so Meseguer's Lemma applies to **Set**. We show the necessity of Lemma 4.2.6's hypothesis that a map $a_{\lambda+1}$ such that $a_{\lambda+1} m_\lambda = a_\lambda(F_\lambda a_\lambda)$ can be defined. The section discusses a subfunctor of the free-magma monad, which has an algebra that cannot extend to a monad algebra. A *magma* is a set with a binary operation, subject to no equations.

The free magma monad (T, η, μ) has the following parts:

1. Given a set X, TX consists of all words with variables in X. A *magma word* is any expression formed by finitely many applications of the rules:

(i) If $x \in X$, then $\textcircled{x}$ is a magma word.

(ii) If s, t are magma words, then



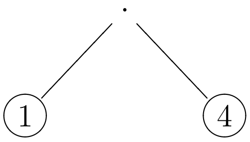
is a magma word – the product of s and t. For ease in reading, we use binary tree notation for products.

2. T defines a functor: given $f : X \to Y$, to compute Tf we apply f to all members of X in a given word, leaving the tree and circle structure unchanged.

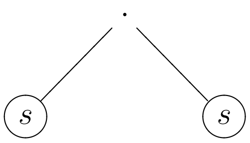
3. The "insertion of variables" map $\eta X : X \to TX$, $(\eta X)(x) = \textcircled{x}$.

4. The "semantic composition" map $\mu X : T^2X \to TX$ sends a word $s \in T^2X$ of words to a word in TX, by removing the circles around each element of TX used in making s. μ is also a natural transformation.

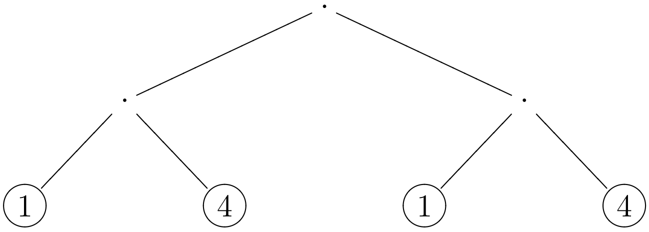
The notation takes a little while to soak in; to expedite the process, we consider a calculation with $T\mathbb{N}$. Suppose $s \in T\mathbb{N}$ is the word



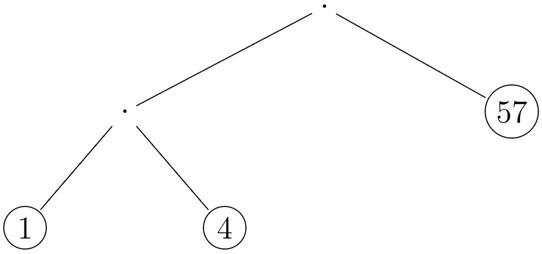
and $\mathbf{t_1} = ss \in T^2(\mathbb{N})$ is the word



then $(\mu\mathbb{N})(\mathbf{t_1})$ is



Now we define a subfunctor F of T. Define the *depth* $dep(n)$ of a node n in a binary tree inductively by: the depth of the $dep(root) = 0$; if a is immediately below b, then the $dep(a) = dep(b) + 1$. Define a *leaf* to be a node that has nothing below it. Let F consist of all rooted, labeled, binary trees (i.e., magma words) such that the depth of each leaf is the same. For example s and $(\mu\mathbb{N})(\mathbf{t_1})$ are in $F(\mathbb{N})$, but



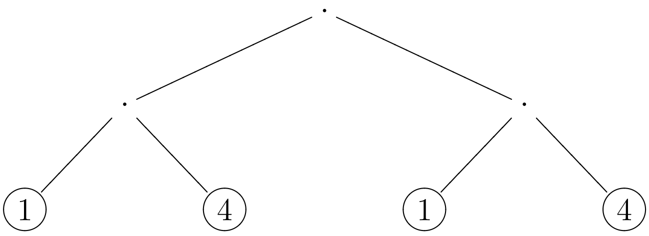
is not, because $dep(57) = 1$ and $dep(1) = dep(4) = 2$. Evidently, F is a subfunctor, and for any X, $(\eta X)^{+1}(X) \subseteq F(X)$. But, for any non-empty X, $(\mu X)(F^2(X))$ is not contained in $F(X)$. One readily verifies that $\bar{F} = T$.

Now we define an partial algebra structure on $\mathbb{N}$ that does not extend to a T-algebra structure. As in Lemma 4.2.6, a structure map for a pair $(F, \eta : \mathrm{id}_{\mathcal{A}} \to F)$ is a map $a : FA \to A$ satisfying $id_A = a(\eta A)$. Define $n : F(\mathbb{N}) \to \mathbb{N}$ on a tree r as follows: if the depth of each leaf of r is odd then $n(r)$ is the leftmost label, if the depth of each leaf of r is even (or zero), then $n(r)$ is the rightmost label. For example, $n(s) = 1$ (s defined above) and $n(\mu\mathbb{N})(\mathbf{t_1}) = 4$.

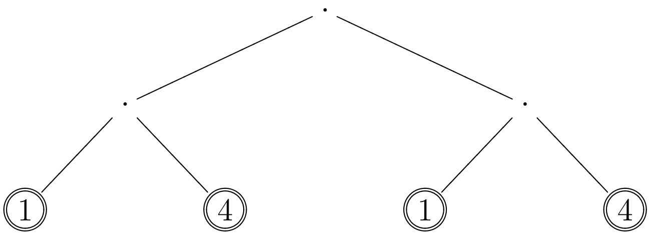
In order for $n : F(\mathbb{N}) \to \mathbb{N}$ to extend to $n : F_1(\mathbb{N}) \to \mathbb{N}$ there must be a function n_1 making

$$\begin{tikzcd}
F^2(\mathbb{N}) \arrow[r, "Fn"] \arrow[d, "\mu\mathbb{N}"'] & F(\mathbb{N}) \arrow[d, "n"] \\
F(\mathbb{N}) \arrow[r, "n_1"] & \mathbb{N}
\end{tikzcd}$$

commute. No such a_1 can exist, for $(\mu\mathbb{N})(\mathbf{t_1})$ is



which equals $(\mu\mathbb{N})(\mathbf{t_2})$, where $\mathbf{t_2}$ is



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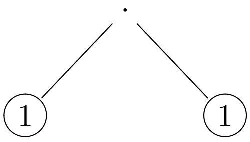
Finally, note that $(Fn)(\mathbf{t_1})$ is



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so $n(Fn)(\mathbf{t_1}) = 1$. Note that $(Fn)(\mathbf{t_2}) = (\mu\mathbb{N})(\mathbf{t_2})$, so $n(Fn)(\mathbf{t_2}) = 4$. Thus we have elements $\mathbf{t_1}, \mathbf{t_2} \in F^2\mathbb{N}$, that $\mu\mathbb{N}$ identifies and $n(Fn)$ does not identify, so there is no function n_1 such that $n_1(\mu\mathbb{N}) = n(Fn)$.

CHAPTER 5
FREE ALGEBRAS

This chapter explores a generalization of complete distributivity for $\mathbb{P}_m^j$-objects. In several steps, the free complete distributive lattice monad is constructed, then Meseguer's Lemmas are used to construct an appropriate submonad, whose algebras generalize completely distributive lattices.

Section 5.1 describes monads for complete semilattices; $\mathfrak{D}$ is the monad for complete join semilattices, $\mathfrak{U}$ is the monad for complete meet semilattices.

The gist of section 5.2 is that there is a lifting of $\mathfrak{U}$ over $\mathfrak{D}$ (and a lifting of $\mathfrak{D}$ over $\mathfrak{U}$). Therefore, $\mathfrak{U}\mathfrak{D}$ and $\mathfrak{D}\mathfrak{U}$ are monads over $\mathbb{P}$. (See results of Beck [5] summarized in Section 3.4.) In Raney [27], it was shown that complete meet distributivity is the same as complete join distributivity. Hence, the composite monads $\mathfrak{U}\mathfrak{D}$ and $\mathfrak{D}\mathfrak{U}$ have the same category of algebras. The objects in either category are complete lattices where meets distribute over joins, and joins distribute over meets.

Completely distributive complete lattices have been thoroughly studied. The basic structure is described in Raney [27], [28], and [29]. Free objects over **Set** were initially described in Markowsky [24]. Tunnicliff [32] discusses properties of the free completely distributive lattice over a poset. Free objects over $\mathbb{P}$ and the relationship between completely distributive lattices and continuous lattices are described in Hoffman and Mislove [14]. The approach here is apparently new, but yields obviously equivalent free objects.

Each pair (j, m) of subset systems gives rise to a subfunctor of $\mathfrak{U}\mathfrak{D}$. Meseguer's Lemmas are applied to this subfunctor to produce a monad $\overline{F}$. Any $\mathbb{P}_m^j$-objects which is $\mathbb{P}^j_m$-embeddable in a completely distributive complete lattice is an $\overline{F}$-algebra. Any $\overline{F}$-algebra has a natural $\mathbb{P}^j_m$ structure. Because of computational difficulties, no exact algebraic characterization of $\overline{F}$-algebras is given here.

A word about notation: the (functor parts) of the monads described below are given by families of sets. Thus, checking the unit and associative laws requires working with many levels of the power set tower. Roman letters $S, T, \cdots$ denote sets. A subscipt designates the "power set complexity": $S_1 \in \mathcal{P}(A)$, $S_2 \in \mathcal{P}^2 A$ – S_2 is a family of sets, $S_3 \in \mathcal{P}^3 A$ – S_3 is a family of families of sets, etc.

5.1 Complete semilattices

In this section, we describe the free complete join (resp. meet) semilattice on a poset A, using the monad $(\mathcal{D}, d, \mu)$ (resp. $(\mathcal{U}, i, \mu)$). The description of $\mathcal{D}$ is well known; for example, Meseguer [25] uses it. The reader will have noticed that μ is used as a name for two different natural transformations; this would normally be horrible notation, but in this case, the formula for μ is the same. Thus, our notational econony should cause no confusion.

The functors $\mathcal{U}$ and $\mathcal{D}$ act on a poset A by

$$\mathcal{U}(A) = \{S \subseteq A : x \geq y \in S \implies x \in S\}$$

– the set of increasing subsets of A ordered by **reverse inclusion**– and

$$\mathcal{D}(A) = \{S \subseteq A : x \leq y \in S \implies x \in S\}$$

–the set of decreasing subsets of A ordered by inclusion.

Given monotone $f : A \to B$, we define

$$\mathcal{D}f : \mathcal{D}(A) \to \mathcal{D}(B) : S \mapsto \{b \in B : \exists s \in S,\, b \leq f(s)\}$$

and

$$\mathcal{U}f : \mathcal{U}(A) \to \mathcal{U}(B) : S \mapsto \{b \in B : \exists s \in S,\, b \geq f(s)\}$$

The following facts will be of later use.

1. $\mathcal{U}$ and $\mathcal{D}$ are functors: trivially they respect identity arrows. Observe that for any $S \in A$ and montone functions $f : A \to B$, $g : B \to C$,

   $$\begin{aligned} \mathcal{D}(gf)(S) &= \Big\{c \in C : \exists s \in S,\, c \leq gf(s)\Big\} \\ &= \Big\{c \in C : \exists f(s) \in \mathcal{D}f(S),\, c \leq gf(s)\Big\} \\ &= (\mathcal{D}g)(\mathcal{D}f) \end{aligned}$$

   and, similarly, $\mathcal{U}(gf)(S) = (\mathcal{U}g)(\mathcal{U}f)(S)$.

2. For each poset A, $\mathcal{D}(A)$ is a complete lattice, with supremum operation given by set theoretic union, and infimum given by intersection. $\mathcal{U}(A)$ is a complete lattice, with infimum given by union and supremum given by intersection.

3. For each $S_2 \subseteq \mathcal{D}(A)$,

   $$\mathcal{D}f\Big(\bigcup S_2\Big) = \bigcup\Big\{(\mathcal{D}f(S)) : S \in S_2\Big\}.$$

   In particular, this holds if $\mathfrak{S}$ is empty. So $\mathcal{D}f$ preserves all suprema. Similarly, if $S_2 \subseteq \mathcal{U}(A)$, then

   $$\mathcal{U}f\Big(\bigcup \mathfrak{S}\Big) = \bigcup\Big\{(\mathcal{U}f)(S) : S \in \mathfrak{S}\Big\}$$

   so that $\mathcal{U}f$ preserves all infima.

Define

$dA : A \to \mathcal{D}(A) : a \mapsto \{x \in A : x \leq a\}$,

$iA : A \to \mathcal{U}(A) : a \mapsto \{x \in A : x \geq a\}$, and

$\mu A : \mathcal{D}^2(A) \to \mathcal{D}(A) : S_2 \in \mathcal{D}^2(A) \mapsto \bigcup S_2$.

It may be puzzling that $\mathcal{D}(A)$ is ordered by subset inclusion and $\mathcal{U}(A)$ is ordered by reverse subset inclusion. The fact that

$$x \leq y \iff dA(x) \subseteq dA(y) \iff iA(y) \subseteq iA(x)$$

motivates the choice, because one wants iA to be order preserving. Moreover, one wants $\mathcal{U}f$ to preserve all infima, which only happens if the infimum operation in $\mathcal{U}A$ is union.

The reader may check that i, d and μ are natural transformations. The monad assocative law

$$\begin{tikzcd}
\mathcal{D}^3 \arrow[r, "\mathcal{D}\mu"] \arrow[d, "\mu\mathcal{D}"] & \mathcal{D}^2 \arrow[d, "\mu"] \\
\mathcal{D}^2 \arrow[r, "\mu"] & \mathcal{D}
\end{tikzcd}$$

holds for both $\mathcal{D}$ and $\mathcal{U}$, because if $S_3 \in \mathcal{P}^3(A)$, then

$$\bigcup\bigcup S_3 = \bigcup\Big\{\bigcup S_2 : S_2 \in S_3\Big\}.$$

The monad unit laws

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\mathcal{D}d"] \arrow[dr, equal] & \mathcal{D}^2 \arrow[d, "\mu"] & \mathcal{D} \arrow[l, "d\mathcal{D}"'] \arrow[dl, equal] \\
 & \mathcal{D} &
\end{tikzcd}$$

hold for $\mathcal{D}$, because if $S \in \mathcal{D}(A)$, then

$$S = \bigcup\Big\{(dA)(x) : x \in S\Big\} = \bigcup\Big\{T \in \mathcal{D}(A) : T \subseteq S\Big\}.$$

Similar computation shows the unit laws hold for $\mathcal{U}$.

Now to describe algebras over these monads:

Lemma 5.1.1. *Let $T \subseteq \mathcal{D}(A)$, such that for all $x \in A$, $(dA)(x) \in T$. Then the following are equivalent:*

1. *there is an order preserving map $\alpha : T \to A$ such that $\mathrm{id}_A = \alpha(dA)$;*
2. *each $S \in T$ has a supremum;*
3. *$dA : A \to \mathcal{D}A$ has a right adjoint.*

The analogous conditions involving $T \subseteq \mathcal{U}A$ are also equivalent.

Proof. We give the proof for $T \subseteq \mathcal{D}A$, leaving the upside-down argument for $T \subseteq \mathcal{U}A$ to the reader.

($2 \implies 1$) Define $\alpha(S) = \bigvee S$ and compute.

($1 \implies 2$) $\alpha(S)$ is an upper bound for S, because if $x \in S$, then $(dA)(x) \subseteq S$, whence $x = \alpha(dA)(x) \leq \alpha(S)$. Suppose for all $x \in S$, $x \leq u$. Then $(dA)(x) \subseteq (dA)(u)$, so $S = \bigcup\{(dA)(x) : x \in S\} \subseteq (dA)(u)$, therefore $\alpha(S) \leq u$.

($2 \iff 3$) By definition of $\bigvee S$, $\downarrow x \subseteq S \iff x \leq \bigvee S$. ☐

Proposition 5.1.2. $\mathbb{P}^\infty$ *is equivalent to the category of* $\mathcal{D}$*-algebras.* $\mathbb{P}_\infty$ *is equivalent to the category of* $\mathcal{U}$*-algebras.*

Proof. The first assertion is proved, leaving the second to the reader. By Lemma 5.1.1, any $\mathcal{D}$-algebra has a supremum for each decreasing set; the supremum of an arbitrary set S is equal to the supremum of the decreasing set $\downarrow S$. By the definition $\mathcal{D}$-algebra homomorphisms coincide with order preserving functions which preserve all suprema of decreasing sets. It is easy to see that a map preserves suprema of all decreasing sets if and only if it preserves suprema of all sets.

Conversely, if $A \in \mathbb{P}^\infty$ it has structure map

$$\alpha : \mathcal{D}(A) \to A : S \mapsto \bigvee S.$$

In Lemma 5.1.1 it was noted that α satisfies the unit law for structure maps. α satisfies the associative law by the order theoretic fact proved in Lemma 5.1.3 below. ☐

Lemma 5.1.3. *For any* $S_2 \in \mathcal{D}^2(A)$,

$$\bigvee \left\{ x \in A : \exists S_1 \in S_2,\, x \leq \bigvee S_1 \right\} = \bigvee \bigcup S_2.$$

Proof. To verify this equality, one notes that for all $x \in \bigcup S_2$, $\exists S_1 \in S_2$ such that $x \in S_1$, so $x \leq \bigvee S_1$. Thus the left hand side dominates the right. If u is an upper bound for $\bigcup S_2$, and $\exists S_1 \in S_2$, $x \leq \bigvee S_1$, then $x \leq u$. So the left hand side is the least upper bound of $\bigcup \mathfrak{S}$; this establishes the lemma. ☐

5.2 Completely Distributive Complete Lattices

In the following arguments, the forgetful functors are supressed from the notation. The lemmas that follow establish that there is a lifting of $\mathcal{U}$ over $\mathcal{D}$ (and similar arguments show that there is a lifting of $\mathcal{D}$ over $\mathcal{U}$). Explicitly this means:

1. For any $\mathcal{U}$-algebra A, there is a $\mathcal{U}$-structure on $\mathcal{D}(A)$.
2. $\mathcal{U}$ maps $\mathcal{D}$-algebra maps to $\mathcal{D}$-algebra maps, so we may view $\mathcal{U}$ as a functor $\mathbb{P}^{\mathcal{D}} \to \mathbb{P}^{\mathcal{D}}$.
3. Both natural transformations i and $\mu : \mathcal{U}^2 \to \mathcal{U}$ preserve all joins; similarly, d and $\mu : \mathcal{D}^2 \to \mathcal{D}$ preserve all meets.

Lemma 5.2.1. *For any poset A, both $\mathcal{U}(A)$ and $\mathcal{D}(A)$ are complete lattices. Thus, $\mathcal{U}(A)$ and $\mathcal{D}(A)$ are both $\mathcal{U}-$ and $\mathcal{D}-$ algebras.*

Lemma 5.2.2. *For any poset A,*

1. $dA : A \to \mathcal{D}(A)$ prserves all existing infima.

2. $iA : A \to \mathcal{U}(A)$ prserves all existing suprema.

Proof. Suppose $S \subseteq A$ and $u = \bigwedge S$ exists. Since dA is order preserving ,

$$(dA)(u) \subseteq \bigcap \Big\{ dA(x) : x \in S \Big\}.$$

For the reverse inequality, suppose $\ell \in \bigcap\{dA(x) : x \in S\}$, i.e., ℓ is a lower bound of S. Then $\ell \leq u$, so $\ell \in dA(u)$. The proof for uA is upside-down, but otherwise identical. □

Lemma 5.2.3. *If $f : A \to B$ preserves all suprema, then $\mathcal{U}f : \mathcal{U}(A) \to \mathcal{U}(B)$ also preserves all suprema. Similarly, if f preserves infima, so does $\mathcal{D}(f)$. Therefore, for any order preserving map f, both $\mathcal{U}\mathcal{D}f$ and $\mathcal{D}\mathcal{U}f$ preserve all infima and suprema.*

Proof. A proof of the first fact is given, the second is very similar, but notationally easier. Use the fact that $\mathcal{U}(A)$ is a complete sublattice of $\mathcal{P}(A)$, and therefore completely distributive.

Let $(S_\lambda)_{\lambda \in L}$ be an indexed subset of $\mathcal{U}(A)$; for each S_λ, choose a family $(x_{\lambda,\kappa} : \kappa \in K)$ such that

$$S_\lambda = \uparrow (x_{\lambda,\kappa} : \kappa \in K).$$

Recall that:

- joins in $\mathcal{U}A$ are intersections;
- iA preserves suprema, which reads

  $$\uparrow \bigvee y_\lambda = (iA)(\bigvee y_\lambda) = \bigcap_\lambda \uparrow y_\lambda,$$

  for any indexed family $(y_\lambda) \subseteq A$.

One computes as follows:

$$\begin{aligned}
\mathcal{U}f\Big(\bigcap_\lambda S_\lambda\Big) &= \mathcal{U}f\Big(\bigcap_\lambda \bigcup_\kappa \downarrow x_{\lambda,\kappa}\Big) \\
&= \mathcal{U}f\Big(\bigcup \Big\{ \bigcap \uparrow x_{\lambda,c(\lambda)} :\ c : L \to K \Big\}\Big) \\
&= \bigcup \mathcal{U}f\Big(\Big\{ \bigcap \uparrow x_{\lambda,c(\lambda)} :\ c : L \to K \Big\}\Big) \\
&= \bigcup \Big\{ \uparrow f\Big(\bigvee_\lambda x_{\lambda,c(\lambda)}\Big) :\ c : L \to K \Big\} \\
&= \bigcup \Big\{ \uparrow \bigvee f(x_{\lambda,c(\lambda)}) :\ c : L \to K \Big\} \\
&= \bigcup \Big\{ \bigcap_\lambda \uparrow f(x_{\lambda,c(\lambda)}) :\ c : L \to K \Big\} \\
&= \bigcap_\lambda \mathcal{U}f(S_\lambda)
\end{aligned}$$

□

Corollary 5.2.4. *For any poset A,*

1. $\mu : \mathcal{D}^2(A) \to \mathcal{D}(A)$ *prserves all infima.*
2. $\mu : \mathcal{U}^2(A) \to \mathcal{U}(A)$ *prserves all suprema.*

Proof. Suppose $S_{2,\lambda} \in \mathcal{D}^2(A)$, for $\lambda \in L$: claim 1 amounts to

$$\bigcup \bigcap_{\lambda} S_{2,\lambda} = \bigcap_{\lambda} \bigcup S_{2,\lambda}.$$

Clearly, the left hand side is contained in the right. Suppose that x is a member of the right hand side, that is, for each λ there exists $S_{1,\lambda} \in S_{2,\lambda}$ with $x \in S_{1,\lambda}$. Then

$$S_1 = \bigcap_{\lambda} S_{1,\lambda}$$

is in each $S_{2,\lambda}$ because each $S_{2,\lambda}$ is downward closed – relative to the inclusion order in $\mathcal{D}^2(A)$. Since $x \in S_1$, this proves that x is in the right hand quantity, and the desired equality of sets holds.

The proof for the $\mathcal{U}$ works similarly. □

The preceding lemmas show that there are liftings of $\mathcal{U}$ over $\mathcal{D}$ (and vice versa), so we have:

Corollary 5.2.5. *$\mathcal{UD} = (\mathcal{UD}, \eta, \nu)$, where*

$$\eta : \mathrm{id}_{\mathbb{P}} \to \mathcal{UD} : a \mapsto \{S \in \mathcal{D}A : \downarrow a \subseteq S\}$$

and

$$\nu : \mathcal{UDUD} \to \mathcal{UD} : S_4 \mapsto \bigcup \bigcap S_4,$$

is a monad over $\mathbb{P}$ whose algebras are completely distributive complete lattices.

Proof. The only thing left to be proved is the formulas for the natural transformations. By Beck's reasoning – as summarized in Section 3.4, η is the horizontal composition $i \cdot d$. To compute $\nu : \mathcal{UDUD} \to \mathcal{UD}$ one uses the fact that νA is the structure map for the free algebra $\mathcal{UD}A$. Consider the situation in light of the discussion following Theorem 3.4.5. (Here $\mathcal{U}$ plays the role of S, while $\mathcal{D}$ plays the role of T.) The relation $a = a_T(Ta_S)$ applied to the object $\mathcal{UD}A$ implies

$$\begin{aligned}(\nu A)(S_4) &= \bigcup \left(\mathcal{D} \bigcap\right) \left(S_4\right) \\ &= \bigcup \left\{T \in \mathcal{D}A : \exists S_2 \in \bigcap S_4,\, T \subseteq \bigcap S_2\right\} \\ &= \bigcup \bigcap S_4.\end{aligned}$$

The assertion about algebras follows from Theorem 3.4.5. □

Remark 5.2.6. $\mathcal{UD}$ is a monad because there is a lifting of $\mathcal{U}$ over $\mathbb{P}^{\mathcal{D}}$. One could use the correspondence between liftings and distributive laws outlined in 3.4 to find a natural transformation $\mathcal{DU} \to \mathcal{UD}$. This distributive law is not needed for the calculations which follow, and is somewhat cumbersome, so its explicit description is omitted.

5.3 Some categories of algebras

In this section, Meseguer's Lemmas are applied to the monads described above. Note that $\mathbb{P}$ – the category of posets and order preserving maps – is a cocomplete SF-category in which unions of chains are colimits. Note that extremal monomorphisms in $\mathbb{P}$ are inclusions of subsets with the induced order: each order preserving map $f : A \to B$ factors as $A \to f(A) \to B$, where $f(A)$ is the set-theoretic image of A, with the order induced from B. Thus, the results of Section 4.2 do apply to $\mathbb{P}$.

The reader is advised to review the definition of subset system, given in 1.2.1, if necessary.

Assume all subset systems Z are non-trivial in the sense that for each A, and $a \in A$, $\{a\} \in Z(A)$; this does not reduce the generality of the argument, because given any subset system, one may adjoin all singletons to it without changing which optimum bounds are preserved.

Theorem 5.3.1. [25] *$\mathbb{P}^j$ and $\mathbb{P}_m$ are monadic.*

Proof. Given a subset system j

$$J_0(A) = \{\downarrow S : S \in j(A)\}$$

defines a subfunctor of $\mathcal{D}$. This extends to a subfunctor $\overline{j}$ of $\mathcal{D}$ such that $(\overline{j}, \overline{d}, \overline{\mu})$ is a monad. The natural transformations $\overline{d}$ and $\overline{\mu}$ are obtained from d and μ by suitably modifying the domains and codomains. The precise definition of these maps is contained in the proof of Lemma 4.2.6.

If (A, a_0) has a J_0-structure, Lemma 5.1.1 shows that

$$a_0(S) = \bigvee S$$

for all $S \in J_0(A)$. Thus, to define a_1 one must show some map $J_1 A \to A$ makes the following diagram commute.

$$\begin{tikzcd}
J_0^2 A \arrow[r, "J_0(\bigvee)"] \arrow[d, "\bigcup"] & J_0 A \arrow[d, "\bigvee"] \\
J_1 A \arrow[r] & A
\end{tikzcd}$$

Lemma 5.1.3 shows that $\bigvee S$ is defined for all $S \in J_1 A$ and, moreover, that for $S_2 \in J_0^2 A$,

$$\bigvee \bigcup S_2 = \bigvee \left(J_0 \bigvee \right) \left(S_2 \right).$$

Identical arguments show show that for each λ, if

$$a_\lambda(S) = \bigvee S$$

that

$$a_{\lambda+1}(S) = \bigvee S.$$

It follows that any poset, A, in which each member of $J_0 A$ has a supremum, each member of $\overline{J}(A)$ has a supremum. Moreover, each map preserving J_0-suprema also preserves J-suprema. Thus, one obtains an equivalence of categories between $\mathbb{P}^j$ and $\mathbb{P}^{\overline{J}}$.

An upside-down version of this argument shows that $\mathbb{P}_m$ is a category of algebras. $\square$

Given subset systems m and j, $F := M_0 J_0$ forms a subfunctor of

$$T := \mathfrak{U}\mathfrak{D}.$$

By Corollary 5.2.5, (T, ν, η) is a monad on $\mathbb{P}$. By 4.2.1, there exists a smallest $\overline{F}$ above F that is closed under ν. This monad is used to discuss the categories defined below.

Definition and Remarks 5.3.2. Let $\mathbb{DP}^\infty_\infty$ denote the category of completely distributive complete lattices with maps, which preserve order and all optimum bounds. $\mathbb{DP}^j_m$ denotes the full subcategory of $\mathbb{P}^j_m$ containing objects such that there is a $\mathbb{P}^j_m$ map $\phi : A \to B$, with ϕ $\mathbb{P}$-extremal mono, and $B \in \mathrm{Obj}(\mathbb{DP}^\infty_\infty)$.

$\mathbb{M}^j_m$ denotes the category of $\overline{F}$-algebras, where $\overline{F}$ is the monad described above.

It will be shown that

$$Sp\mathbb{P}^j_m \subseteq \mathbb{DP}^j_m \subseteq \mathbb{M}^j_m.$$

To see the first inclusion, one notes that any family $\mathcal{F}$ of subsets of a set X which is also a $\mathbb{P}^j_m$ object is in $\mathbb{DP}^j_m$; the inclusion $\mathcal{F} \subseteq \mathcal{P}(X)$ is a $\mathbb{P}^j_m$-embedding and $\mathcal{P}(X)$ is a completely distributive complete lattice. Proposition 5.3.7 shows the second inclusion.

Let $A \in \mathbb{DP}^j_m$, and $\phi : A \to B$ be given as in the definition of $\mathbb{DP}^j_m$; let $b : TB \to B$ be the T-structure for B. The verification of the hypothesis of 4.2.6 proceeds by comparing the diagram to be completed with the $\mathcal{UD}$-algebra associativity diagram for B. From this point on, the notation of Lemma 4.2.2 is adopted with slight modification; $\nu : T^2 \to T$ is the monad multiplication; $\nu_\lambda : F^2_\lambda \to F_{\lambda+1}$ is defined inductively (and plays the role that was played by m_λ in the proof of 4.2.2); $\eta : \mathrm{id}_\mathbb{P} \to T$ is the unit; $\eta_\lambda : \mathrm{id}_\mathbb{P} \to F_\lambda$ are defined inductively, and play the role that was played by n_λ in 4.2.2. The natural transformations $f_\lambda : F_\lambda \to T$ and $c_\lambda : F_\lambda \to F_{\lambda+1}$ play the same roles as in Lemma 4.2.2.

Remark 5.3.3. By Lemma 3.4.5, the structure map $b : TB \to B$ is given by:

$$b(S_2) = \bigwedge \Big\{ \bigvee S_1 : S_1 \in S_2 \Big\}.$$

Since ϕ is a $\mathbb{P}^j_m$-map, the following diagram commutes.

$$\begin{tikzcd}
FA \arrow[r, "F\phi"] \arrow[d, "a_0"] & FB \arrow[r, "f_\lambda B"] \arrow[d] & TB \arrow[d, "b"] \\
A \arrow[r, "\phi"] & B \arrow[r, equal] & B
\end{tikzcd}$$

The left square expresses that ϕ commutes with the F-structure; the right square expresses that the F-structure on B is the restriction of the T-structure.

Lemma 5.3.4. *Define* $a_0 : FA \to A$ *by* $a_0(S_2) = \bigwedge(\mathcal{U}\bigvee)(S_2)$*; this map satisfies* $a_0\eta_0 = \mathrm{id}_A$*.*

Proof. Let us begin by more explicitly calculating a_0. Using the definition of $\mathcal{U}$ on maps we find that

$$\begin{aligned} a_0(S_2) &= \bigwedge\left(\mathcal{U}\bigvee\right)\left(S_2\right) \\ &= \bigwedge\left\{x \in A : \exists S_1 \in S_2, \bigvee S_1 \leq x\right\} \\ &= \bigwedge\left\{\bigvee S_1 : S_1 \in S_2\right\}. \end{aligned}$$

This definition makes sense; each $S_1 \in J_0(A)$ so each $\bigvee S_1$ exists because $A \in \mathbb{P}^j_m$. $(\mathcal{U}\bigvee)$ is an order preserving map, and $S_2 \in m_0 j_0(A)$, which implies that

$$(\mathcal{U}\bigvee)(S_2) \in M_0(A).$$

Thus, the meet defining $a_0(S_2)$ exist.

Recall $\eta(x) = \{S \in \mathcal{D}A : \downarrow x \subseteq S\}$. Now one calculates

$$\begin{aligned} a_0(\eta(x)) &= \bigwedge\{\bigvee S : S \in \mathcal{D}A, \downarrow x \subseteq S\} \\ &= x \end{aligned}$$

□

For later calculations, it is crucial to know that some of the maps are epi or mono.

Lemma 5.3.5. *For each ordinal* λ*,* $\nu_\lambda A$ *is surjective. Therefore* $\nu_\lambda A$ *is* $\mathbb{P}$*-epi.*

Proof. Because of the structure of $\mathbb{P}$, unions of subposets actually are set theoretic unions. Let $S \in F_{\lambda+1} = F_\lambda \cup (\nu_\lambda A)^{+1}(F^2_\lambda A)$. If $S \in (\nu_\lambda A)^{+1}(F^2_\lambda A)$, then there is an $S_2 \in F^2_\lambda A$ such that $(\nu_\lambda A)(S_2) = S$. Since $c_\lambda A = (\nu_\lambda A)(\eta_\lambda F_\lambda A)$ so for any $S_1 \in F_\lambda A$, $T_2 := (\eta_\lambda F_\lambda A)(S)$ is a member of $F^2_\lambda A$ such that $(\nu_\lambda A)(T_2) = S_1$.

This lemma holds in any category where unions of subobjects actually are set-theoretic unions. □

After noting that extremal monomorphisms in $\mathbb{P}$ are injections $f : A \to B$, where $f(A)$ has the order induced as a subset of B, one easily verifies:

Lemma 5.3.6. *The functors $\mathcal{U}$, $\mathcal{D}$, T, and F_λ all preserve extremal monomorphisms.*

Now one can verify the hypothesis of 4.2.6 is satisfied.

Proposition 5.3.7. *Let $A \in \mathrm{Obj}(\mathbb{DP}^j_m)$, $B \in \mathrm{Obj}(\mathbb{DP}^\infty_\infty)$, $\phi : A \to B$ as in the definition of $\mathbb{DP}^j_m$. Then*

(ex) for each λ, there is a unique map

$$a_{\lambda+1} : F_{\lambda+1}A \to A,$$

such that

$$a_\lambda(F_\lambda a_\lambda) = a_{\lambda+1}\nu_\lambda;$$

(com) for each λ, $\phi a_\lambda = b(f_\lambda B)(F_\lambda \phi)$.

Thus, $\mathbb{DP}^j_m \subseteq \mathbb{M}^j_m$

Proof. Remark 5.3.3 established (com) for $\lambda = 0$. If (com) holds for sucessor ordinals, properties of unions insure that (com) holds at all ordinals.

Suppose (com) holds at stage λ; we will establish (com) and (ex) hold for stage $\lambda + 1$. Consider the following diagram.

$$\begin{tikzcd}
T^2B \arrow[rrrrr, "Tb"] \arrow[ddddd, "\nu B"'] & & & & & TB \arrow[ddddd, "b"] \\
& F_\lambda^2 B \arrow[ul, "f_\lambda^2 B"'] \arrow[rrr, "(F_\lambda b)(F_\lambda f_\lambda B)"] \arrow[ddd, "\nu_\lambda B"] & & & F_\lambda B \arrow[ur, "f_\lambda B"] \arrow[ddd, "b(f_\lambda B)"] & \\
& & F_\lambda^2 A \arrow[ul, "F_\lambda^2 \phi"'] \arrow[r, "F_\lambda a_\lambda"] \arrow[d, "\nu_\lambda A"] & F_\lambda A \arrow[ur, "F_\lambda \phi"] \arrow[d, "a_\lambda"] & & \\
& & F_{\lambda+1} A \arrow[dl, "F_{\lambda+1}\phi"] & A \arrow[dr, "\phi"] & & \\
& F_{\lambda+1} B \arrow[dl, "f_{\lambda+1} B"] & & & B \arrow[dr, equal] & \\
TB \arrow[rrrrr, "b"] & & & & & B
\end{tikzcd}$$

The diagram commutes: the outer square commutes because of the T-algebra associative law for B; the left outer trapezoid commutes because of the definition of $\nu_\lambda B$; commutativity of the right outer trapezoid is obvious; the left inner trapezoid expresses naturality of ν_λ; the top and right inner trapezoids commute by (com).

Establishing commutativity of the top outer trapezoid requires a bit more. Since $f_\lambda : F_\lambda \to T$ is natural, $(Tb)(f_\lambda TB) = (f_\lambda B)(F_\lambda b)$. By definition of horizontal composition

$$f_\lambda^2 B = (f_\lambda TB)(F_\lambda f_\lambda B).$$

Thus,

$$\begin{aligned}
(Tb)(f_\lambda^2 B) &= (Tb)(f_\lambda TB)(F_\lambda f_\lambda B) \\
&= (f_\lambda B)(F_\lambda b)(F_\lambda f_\lambda B)
\end{aligned}$$

This proves that the top outer square commutes.

Consider the (epi,extremal mono)-factorization of

$$b(f_{\lambda+1}B)(F_{\lambda+1}\phi);$$

say it factors as

$$F_{\lambda+1}A \xrightarrow{\alpha} A' \subseteq B.$$

Since $\nu_\lambda A$ is epi, the (epi, extremal mono)-factorization of

$$q := b(f_{\lambda+1}B)(F_{\lambda+1}\phi)(\nu_\lambda A)$$

is

$$F^2_\lambda A \xrightarrow{\alpha(\nu_\lambda A)} A' \subseteq B.$$

The proof of Lemma 4.2.6 shows that each existing a_λ is split epi, which implies both a_λ and (Fa_λ) are epi. Since

$$q = \phi a_\lambda(F_\lambda a_\lambda),$$

the (epi, extremal mono)- factorization of q is also:

$$F^2_\lambda A \xrightarrow{a_\lambda(Fa_\lambda)} A \subseteq B.$$

Uniqueness of factorization implies there is an isomorphism $i : A \to A'$ compatible with the factorizations. So $a_{\lambda+1} = i\alpha$ is a map making (com) hold for $\lambda + 1$.

Reading off the diagram one finds that:

$$\begin{aligned}
\phi a_\lambda(Fa_\lambda) &= b(Tb)(f^2_\lambda B)(F^2_\lambda \phi)\\
&= b(\nu B)(f^2_\lambda B)(F^2_\lambda \phi)\\
&= b(f_{\lambda+1}B)(F_{\lambda+1}\phi)(\nu_\lambda A)\\
&= \phi a_{\lambda+1}(\nu_\lambda A)
\end{aligned}$$

The last step used that (com) holds for $\lambda + 1$. Since ϕ is a monomorphism, one concludes that (ex) is satisfied for $\lambda + 1$.

$a_{\lambda+1}$ is the only map which makes the inner square commute because $\nu_\lambda A$ is epi, by Lemma 5.3.5. □

Corollary 5.3.8. *$\mathbb{DP}^j_m$ is a full subcategory of $\mathbb{M}^j_m$.*

Proof. By Lemma 5.3.3, any $\mathbb{P}^j_m$-map preserves the F-structure of a $\mathbb{DP}^j_m$ object. Lemma 4.2.7 implies that any $\mathbb{P}^j_m$-map extends to a $\mathbb{M}^j_m$ map. ☐

Corollary 5.3.9. *The forgetful functor $\mathbb{DP}^j_m \to \mathbb{P}$ has a left adjoint.*

Proof. First note that $\overline{F}A \in \mathrm{Obj}(\mathbb{DP}^j_m)$ for any A, because $\overline{f}A : \overline{F}A \to TA$ is an embedding of FA in a completely distributive lattice. Because $\overline{F}A$ is the free $\overline{F}$-algebra on A, if $f : A \to B$ is an order preserving map from $A \in \mathrm{Obj}(\mathbb{P})$ to $B \in \mathrm{Obj}(\mathbb{DP}^j_m) \subseteq \mathrm{Obj}(\mathbb{P}^{\overline{F}})$, then there is a $\mathbb{DP}^j_m$-unique map $f^* : \overline{F}A \to B$. This proves that $\overline{\eta}$ has the universal property described in 2.5.2.3; thus, $\overline{F}$ is the left adjoint to the forgetful functor $\mathbb{DP}^j_m \to \mathbb{P}$. ☐

Corollary 5.3.10. *The forgetful functors $\mathbb{DP}^\infty_\infty \to \mathbb{DP}^j_m$ and $\mathbb{DP}^\infty_\infty \to \mathbb{M}^j_m$ have left adjoints.*

Proof. Apply Proposition 3.3.6 and Corollary 6.2.8. (Note: the proof of 6.2.8 does not depend on the arguments in this section, so the result is listed here. There is no circular argument.) ☐

There is some, rather limited, information about members of $\mathbb{M}^j_m$.

Proposition 5.3.11. *If $A \in \mathrm{Obj}(\mathbb{M}^j_m)$, then $A \in \mathrm{Obj}(\mathbb{P}^j_m)$.*

Proof. Define JA, MA, α, β, γ, and δ, by the requirement that the squares below are pullbacks.

$$\begin{tikzcd}
JA \arrow[r, "\beta"] \arrow[d, "\alpha"] & \overline{F}A \arrow[d, "\overline{f}A"] \\
\mathfrak{D}A \arrow[r, "i\mathfrak{D}A"] & \mathfrak{U}\mathfrak{D}A
\end{tikzcd}$$

$$\begin{tikzcd}
MA \arrow[r, "\delta"] \arrow[d, "\gamma"] & \overline{F}A \arrow[d, "\overline{f}A"] \\
\mathfrak{U}A \arrow[r, "\mathfrak{U}dA"] & \mathfrak{U}\mathfrak{D}A
\end{tikzcd}$$

By assumption, $j(A)$ and $m(A)$ contain singletons, therefore

$$(i\mathfrak{D}A)^{+1}(J_0A) \subseteq M_0J_0A \subseteq \overline{F}A,$$

and

$$(\mathfrak{U}dA)^{+1}(M_0A) \subseteq M_0J_0A \subseteq \overline{F}A.$$

It follows that

$$J_0A \subseteq JA = (i\mathfrak{D}A)^{-1}\overline{F}A$$

and

$$M_0A \subseteq MA = (\mathfrak{U}dA)^{-1}\overline{F}A$$

Again using the fact that j and m contain singletons,

$$(dA)^{+1}(A) \subseteq JA$$

and

$$(iA)^{+1}(A) \subseteq MA,$$

so corestrictions $dA| : A \to JA$ and $iA| : A \to MA$ satisfying $\alpha(dA)| = dA$ and $\gamma(iA)| = iA$ exist.

If $a : \overline{F}A \to A$ is an $\overline{F}$-algebra, then $a\beta(dA|) = \mathrm{id}_A$ and $a\delta(iA|) = \mathrm{id}_A$. Lemma 5.1.1 shows that for any subset $\mathcal{F} \subseteq \mathfrak{D}A$ containing each principal down-segment $\downarrow x$, an order preserving map $y : \mathcal{F} \to A$ satisfies $y(dA|) = \mathrm{id}_A$ if, and only if, $y(S) = \vee S$ for each $S \in \mathcal{F}$ (and analogously for subsets of $\mathfrak{U}A$). Thus, we conclude $a\beta = \bigvee$ and $a\delta = \bigwedge$. $\square$

Remark 5.3.12. If $j = m = \omega$ – the subset system which selects all finite subsets, then $Sp\mathbb{P}_\omega^\omega = \mathbb{D}\mathbb{P}_\omega^\omega$. For general j and m, the author does not know if equality holds, but he suspects that the equality does not always hold.

Remark 5.3.13. A fundamental difficulty working with $\mathbb{M}_m^j$ is that one does not immediately know any order theoretic formula for the $\overline{F}$-structure maps. One might hope that each $\overline{F}$-structure map is $S_2 \mapsto \bigwedge(\mathfrak{U}\bigvee)S_2$, but the author cannot presently substantiate such hopes.

Remark 5.3.14. The author knows no examples of objects in $\mathrm{Obj}(\mathbb{M}^j_m) \setminus \mathrm{Obj}(\mathbb{DP}^j_m)$. In light of Corollary 5.3.10, any $\mathbb{DP}^j_m$ object is freely embedded in a completely distributive lattice. In fact, $A \in \mathrm{Obj}(\mathbb{M}^j_m)$ is in $\mathrm{Obj}(\mathbb{DP}^j_m)$ if, and only if, the unit of the adjunction mentioned in 5.3.10 is $\mathbb{P}$-extremal mono.

Li[18] explicitly constructs a map $u : P \to ISF(P)$ (not necessarily mono), with the following properties:

- $ISF(P)$ is a complete, completely distributive lattice.
- u preserves designated meets and joins.
- If $f : P \to A$ is a map preserving designated meets and joins, and A is complete, completely distributive, then there is a unique map $f^* : ISF(P) \to A$ such that $f = f^*u$.

To explain the relationship between Li's results and the results here, note:

- Li uses families $SP \subseteq \mathcal{P}P$ and $IP \subseteq \mathcal{P}P$, which are only required to contain singletons. Joins of SP-sets and meets of IP sets are required to be preserved. In this document, the choice of distinguished optimum bounds is made for all posets at once, via a subset system.
- Comparing universal properties, one sees that Li's construction applied to $P \in \mathrm{Obj}(\mathbb{M}^j_m)$ with $SP = j(P)$ and $IP = m(P)$ yields the left adjoint mentioned in 5.3.10.

Unfortunately, it is very difficult to see when u is an embedding. Let $[S]$ denote the smallest downward closed family of A containing S which is closed under SP-joins. Let $\mathcal{F}P$ denote the class of increasing sets, which are closed under IP-meets. Li defines $IS(P)$ to be the familly of all decreasing sets in $\mathcal{F}P$, ordered by $S \preceq T$ if, and only if, there is an indexed chain $(S_i)_{i\in I}$, with $I = \mathbb{Q} \cap [0, 1]$ such that $S_0 = S$, $S_1 = T$ and whenever $i < j$, $S_i \cap [P \setminus S_j] = 0$. The map $u : P \to ISP$ is defined by $u(x) = \{U \in \mathcal{U}A : U \preceq x\}$.

Nonetheless, $\mathbb{DP}^j_m$, it is a fairly nice category. In fact, it is complete.

Proposition 5.3.15. *$\mathbb{DP}^j_m$ is complete.*

Proof. Note that a poset A is a $\mathbb{P}^j_m$ object if and only if there is an order preserving map $a : J_0A \times M_0A \to A$ such that $a(d_0A \times i_0A) = \mathrm{id}_A$.

Suppose $(A_i)_{i\in I}$ is a family of $\mathbb{P}^j_m$ objects and for each $i \in I$,

$$\phi_i : A_i \to B_i$$

is an embedding of A_i into a completely distributive lattice B_i. The proof of Lemma 3.1.3 implies that the poset product $\prod A_i$ is a $\mathbb{P}^j_m$-object. Moreover, the product-induced map $\phi : \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ is an embedding which preserves (j, m)-optimum bounds. (Essentially what is going on is that meets and joins are computed coordinate-by-coordinate.)

If $f, g : A \to B$ are $\mathbb{P}^j_m$ maps, then the proof of Lemma 3.1.3 implies $\mathrm{eq}(f, g)$ is a (j, m)-complete subset of A. If $A \in \mathrm{Obj}(\mathbb{DP}^j_m)$, then A can be (j, m)-embedded into a completely distributive lattice, so $\mathrm{eq}(f, g)$ may also be so embedded.

Thus, any set of objects in $\mathbb{DP}^j_m$ has a product, and any pair of $\mathbb{DP}^j_m$ maps has an equalizer. By Borceux [6, Volume 1, 2.8.1], this implies $\mathbb{DP}^j_m$ is complete. $\square$

CHAPTER 6
COEQUALIZERS

6.1 Epis and Equalizers in $\mathbb{P}$

The results of this section characterize epis and regular epis in the category $\mathbb{P}$ of posets. The presentation and proofs (except Construction 6.1.2 which is discussed in Meseguer [25]) are the work of the author, but the author believes it likely that they are not new. In all statements, A and B are arbitrary posets.

Lemma 6.1.1. *Let $f : A \to B$ be a monotone map. f is epi if and only if f is onto.*

Proof. Since the forgetful functor $\mathbb{P} \to \mathbf{Set}$ is faithful, if f is onto then f is epi. For the converse, suppose $b \in B \setminus f(A)$. Define

$$S_1 = \downarrow (f(A) \cap \downarrow b)$$

and

$$S_2 = \downarrow b.$$

Evidently, both S_1 and S_2 have the same intersection with $f(A)$, but $b \in S_2 \setminus S_1$. Therefore characteristic functions of $B \setminus S_1$ and $B \setminus S_2$ are distinct maps (say

$$c_1, c_2 : B \to 2$$

are respectively the characteristic functions of $B \setminus S_1$ and $B \setminus S_2$) such that $c_1 f = c_2 f$. Both c_i are order preserving, because S_1 and S_2 are decreasing sets. $\square$

Definition and Remarks 6.1.2. The following construction is paraphrased, following Meseguer [25] pp 73-74.

If $f : A \to B$ is any order preserving map, we may define a preorder $\overset{f}{\preceq}$ on A by

$$a_1 \overset{f}{\preceq} a_2 \iff f(a_1) \leq f(a_2).$$

This relation is obviously reflexive and transitive, but –because f may not be injective – may not be anti-symmetric.

Suppose $(A, \leq)$ is a poset and $\preceq$ is a preorder strengthening $\leq$, i.e.,

$$a_1 \leq a_2 \implies a_1 \preceq a_2.$$

Then we have an order preserving map $\alpha : A \to A/\preceq$. (Recall that $A/\preceq$ is the set of equivalence classes

$$\{x \in A : x \preceq a \text{ and } a \preceq x\}$$

partially ordered by $\preceq$.)

Define maps $f_1, f_2 : A \to B_i$ to be equivalent, if there is an isomorphism, $i : B_1 \to B_2$ such that

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[dr, "f_2"] & B_1 \arrow[d, "i"] \\ & B_2 \end{tikzcd}$$

commutes.

One verifies that:

1. The maps $(f : A \to B) \mapsto \overset{f}{\preceq}$ and $(\preceq) \mapsto \alpha : A \to A/\preceq$ are mutually inverse correspondences between

    - the set (modulo equivalence) of surjective maps f with domain A.
    - the set of preorders on A which strengthen $\leq$.

2. For any $f : A \to B$, $f = c\alpha$, where $\alpha : A \to A/\overset{f}{\preceq}$ and $c(\alpha(a)) = f(a)$.

3. Given surjections $f_1, f_2 : A \to B_i$, there is a $c : B_1 \to B_2$ if and only if

$$a_1 \overset{f_1}{\preceq} a_2 \implies a_1 \overset{f_2}{\preceq} a_2.$$

Remark 6.1.3. Note that any intersection of preorders is a preorder. Thus, the class of preorders strengthening $\leq$ is a complete lattice. We say a set S of ordered pairs *generates* $\preceq$ if $\preceq$ is the smallest preordered containing S. Note that if S generates $\preceq$, we can describe $\preceq$ explicitly: $a_0 \preceq a_n$ if

- $a_0 = a_n$, or
- there is a finite sequence $a_1, a_2, \cdots, a_{n-1}$, such that for all i with $0 \leq i \leq n-1$, $(a_i, a_{i+1}) \in S$.

(The first bulleted condition insures that $\preceq$ is reflexive. The second requirement insures that $\preceq$ is transitive; generally, if S is a relation, the relation obtained by applying the second bulleted item only is called the *transitive closure* of S.)

One may verify the following construction using 6.1.2.

Construction 6.1.4. *Let $f, g : A \to B$ be order preserving maps. The coequalizer of f and g is the quotient of $(B, \leq)$ by the smallest preorder containing $\leq$ and $\mathbb{C}(g,h) = \{(gx, hx), (hx, gx) : x \in A\}$.*

Lemma 6.1.5. *Let $f : A \to B$ be a surjective order preserving map. The following are equivalent.*

1. *f is the coequalizer of some pair $g, h : A_0 \to A$.*
2. *There exist a poset A_0, and maps $g, h : A_0 \to A$ such that the preorder $\overset{f}{\preceq}$ on $(A, \leq)$ is generated by $\leq$ and $\mathbb{C}(g,h)$.*
3. *The preorder $\overset{f}{\preceq}$ on $(A, \leq)$ is generated by $\leq$ and $K(f) = \{(x,y) \in B \times B : f(x) = f(y)\}$*
4. *The preorder $\overset{f}{\preceq}$ on $(A, \leq)$ is generated by $\leq$ and some equivalence relation.*

Proof. ($1 \iff 2$) is evident from 6.1.4. ($3 \implies 4$) is trivial, because $K(f)$ is an equivalence relation. We have

$$\mathbb{C}(g,h) \subseteq K(f) \subseteq \overset{f}{\preceq},$$

because $fg = fh$ and

$$(x, y) \in K(f) \iff f(x) \leq f(y) \text{ and } f(y) \leq f(x).$$

Since $\mathbb{C}(g, h)$ and $\leq_A$ generate $\overset{f}{\preceq}$, $K(f)$ and $\leq$ generate $\overset{f}{\preceq}$.

For ($4 \implies 1$), let $\overset{f}{\preceq}$ be generated by $\leq$ and the equivalence relation $E \subseteq A \times A$ with the trivial order. Then the projection maps $\pi_1, \pi_2 : E \to A$, given by

$$\pi_i(x_1, x_2) = x_i$$

are order preserving. The coequalizer of π_1 and π_2 is f. □

Example 6.1.6. Some readers may be surprised by the fact that surjections are not always $\mathbb{P}$-regular epi. An example of this situation is any map $f : 2_{flat} \to 2$ from the (trivially ordered) two point set onto a two-point chain. Any such map f fails condition 3 from Lemma 6.1.5.

6.2 Factorization of Maps Using Preorders

This section modifies the preorder factorization, to give a first approximation to coequalizers in $\mathbb{P}^j_m$.

Definition 6.2.1. Let Z be a subset system. Z is said to *admit congruences* if for all posets A,

$$Z(A \times A) = \{S \in A \times A : \pi_i(S) \in Z(A), i = 1, 2\}.$$

Lemma 6.2.2. *Suppose both j and m are subset systems which admit congruences. Let $(A, \leq)$ be a poset and $\preceq$ be a preorder strengthening $\leq$. Then $\alpha : A \to A/\preceq$ preserves (j, m)-optimum bounds if and only if $\preceq$ is a $\mathbb{P}^j_m$-subobject of $A \times A$.*

Proof. We show that $\alpha(\vee x_i) = \vee\alpha(x_i)$ for any $\{x_i : i \in I\} \in j(A)$. Clearly $\alpha(\vee x_i)$ is an upper bound of $\{\alpha(x_i) : i \in I\}$. Since $\preceq$ is a $\mathbb{P}^j_m$-subobject of $A \times A$, $\{x_i : i \in I\} \in j(A)$ and for all i, $x_i \leq a$,

$$\vee\{(x_i, a) : i \in I\} = (\vee x_i, a)$$

is a member of $\preceq$; $\vee x_i \leq a$. Thus, any upper bound of $\{\alpha(x_i) : i \in I\}$ dominates $\alpha(\vee x_i)$. (The proof for meets is identical, so omitted.)

Conversely, if α preserves j-joins and m-meets, and (for example)

$$\{x_i : i \in I\}, \{y_i : i \in I\} \in j(A)$$

and for all $i \in I$, $x_i \preceq y_i$, then $\vee_i x_i \preceq \vee_i y_i$. So $\preceq$ is a $\mathbb{P}^j_m$-subobject of $A \times A$. ☐

Definition and Remarks 6.2.3. Suppose j and m admit congruences. Let $(A, \leq) \in \mathrm{Obj}(\mathbb{P}^j_m)$. If $\preceq$ is a preorder strengthening $\leq$ and simultaneously a $\mathbb{P}^j_m$-subobject of $A \times A$, then we say $\preceq$ is a (j, m)-*pocongruence*.

Pocongruences of $\mathbb{P}^j_m$-objects behave very much like preorders of posets.

1. The maps $(f : A \to B) \mapsto \overset{f}{\preceq}$ and $(\preceq) \mapsto \alpha : A \to A/\preceq$ are mutually inverse correspondences between
   - the set (modulo equivalence) of surjective maps f, which preserve (j, m)-optimum bounds and have domain A.
   - the set of (j, m)-pocongruences on A.
2. For any $f : A \to B$, $f = c\alpha$, where $\alpha : A \to A/\overset{f}{\preceq}$ and $c(\alpha(a)) = f(a)$. If f preserves (j, m)-optimum bounds, so do c and α.
3. Given (j, m)-optimum bound preserving surjections $f_1, f_2 : A \to B_i$, there is $c : B_1 \to B_2$ if and only if
$$a_1 \overset{f_1}{\preceq} a_2 \implies a_1 \overset{f_2}{\preceq} a_2.$$

Consideration of Lemma 6.2.2 shows that if $f : A \to B$ is surjective and $S \in j(A)$ (resp. $S \in m(A)$) then $f(S)$ has a supremum (resp. infimum) in B. Moreover, sets of the form $f(S)$ for $S \in j(A)$ (resp. $S \in m(A)$) are the only sets whose optimum bounds are guaranteed to exist. Thus, we are motivated to offer the following conditions on subset systems to guarantee surjective (resp. $\mathbb{P}$-regular epi) images of $\mathbb{P}^j_m$ objects are (j, m)-complete.

Definition and Remarks 6.2.4. Let Z be a subset system. We say Z *preserves surjections* if whenever $f : A \to B$ is a poset surjection,

$$Z(B) = \{f(S) : S \in Z(A)\}.$$

Similarly, we say Z preserves *regular epis* if whenever $f : A \to B$ is a poset regular epimorphism,

$$Z(B) = \{f(S) : S \in Z(A)\}.$$

Since poset regular epimorphims are surjective, any Z that preserves surjections also preserves regular epimorphisms.

Any cardinal κ, the associated subset system κ preserves surjections. For if $f : A \to B$ is surjective, then f is split epi (in **Set**). Hence, there is $s : B \to A$ (not necessarily order preserving) such that $\mathrm{id}_B = fs$. Therefore, any set $S \subseteq B$ with cardinality less than κ is the image of some set $s(S) \subseteq A$. The following remark provides an example of a subset system that does not preserve surjections or regular epis. The author currently does not know of any subset system that preserves regular epis without preserving surjections, but it seems likely such a subset system exists.

A more interesting question, which the author also cannot currently answer is: "do there exist subset systems other than cardinals which preserve regular epimorphisms?"

Remark 6.2.5. Let $j = dir$ be the subset system which selects all upward directed subsets of a poset. Let N denote a countable disjoint union of two-point chains. (See drawing below for help visualizing, and to fix notation.)

$$\begin{tikzcd}
t_0 \arrow[dd] & t_1 \arrow[dd] & \cdots \\
 & & \cdots \\
b_0 & b_1 & \cdots
\end{tikzcd}$$

Evidently N is j-join complete.

Consider the function $f : N \to \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \cdots\}$ with the usual order, defined by $f(b_i) = i$ (for $i \geq 0$) and $f(t_{i-1}) = i$ (for $i \geq 1$). Speaking roughly, "f stacks the two point chains." f is regular epi as a map of posets. Moreover, $\overset{f}{\preceq}$ is a $\jmath$-subobject of $N \times N$, because f preserves all existing joins. But $\mathbb{N}$ is not $\jmath$-join complete, because $\mathbb{N}$ is upward directed but has no join!

Because of the pocongruence factorization for $(\jmath, m)$-bound preserving maps (outlined in 6.2.3) , we have the following first approximation to the coequalizer in $\mathbb{P}^{\jmath}_{m}$. The only thing stopping map $\alpha : B \to B/\preceq$ (described below) from actually being a coequalizer in $\mathbb{P}^{\jmath}_{m}$ is that $B/\preceq$ is generally not $(\jmath, m)$-complete.

Construction 6.2.6. *Suppose $f, g : A \to (B, \leq)$ are $\mathbb{P}^{\jmath}_{m}$ maps, and $\mathbb{C}(f, g)$ is defined as in 6.1.4. Let $\preceq$ be the smallest pocongruence containing $\leq$ and $\mathbb{C}(f, g)$. Then $\alpha : B \to B/\preceq$ has the following universal property: if $h : B \to C$ preserves $(\jmath, m)$-optimum bounds and $hf = hg$, then there is a unique map $i : B/\preceq \to C$ such that $h = i\alpha$.*

Since the quotient map $\alpha : B \to B/\preceq$ is a surjection we have –

Corollary 6.2.7. *Continue with the notation from Construction 6.2.6. If $\jmath$ and m preserve surjections, then $B/\preceq$ is $(\jmath, m)$-complete. Thus, $\alpha : B \to B/\preceq$ is the coequalizer of f and g in $\mathbb{P}^{\jmath}_{m}$. In particular, if $\jmath = \kappa$ and $m = \lambda$ are cardinality subset systems, then α is the coequalizer.*

Corollary 6.2.8. *$\mathbb{DP}^{\infty}_{\infty}$ has coequalizers.*

Proof. The preceding shows that any pair $f, g : A \to B \in \mathbb{P}^{\infty}_{\infty}$ has a coequalizer. Suppose B is completely distributive. Since $\alpha : B \to \mathrm{coeq}_{\mathbb{P}^{\infty}_{\infty}}(f, g)$ is a surjection which preserves all meets and joins, $\mathrm{coeq}_{\mathbb{P}^{\infty}_{\infty}}(f, g)$ is completely distributive. In particular if both A and B are completely distributive, then $\mathrm{coeq}_{\mathbb{P}^{\infty}_{\infty}}(f, g)$ is also the coequalizer in $\mathbb{DP}^{\infty}_{\infty}$. ☐

6.3 Factorization of Meetsemilattice maps

Several simplifications occur describing coequalizers in $\mathbb{P}^{\jmath}_{m}$ if the objects have a meet-semilattice structure, i.e., $\overline{m} \geq \omega$.

Lemma 6.3.1. *Let $f : A \to B$ be a meetsemilattice map. f is $\mathbb{P}$-regular epi if and only if f is surjective.*

Proof. Since $\mathbb{P}$-regular epimorphisms are always surjective, one implication is trivial. For the other, suppose $f : A \to B$ is a surjection preserving binary meets. Then

$$x \overset{f}{\preceq} y \iff f(x \wedge y) = f(x).$$

Thus we have the sequence $x, x \wedge y, y$, where $f(x) = f(x \wedge y)$ and $x \wedge y \leq y$. By Lemma 6.1.5, this proves f is regular epi. □

As noted in the proof, the relation $\overset{f}{\preceq}$ is completely described by $K(f)$. If objects have a meetsemilattice structure, one may use congruences (equivalence relations that are simultaneously (j, m)-subalgebras) rather than the more complex pocongruences.

Finally we have:

Theorem 6.3.2. *Suppose $m \geq \omega$ and j and m preserve regular epis. Let $f : A \to B$ be a $\mathbb{P}^j_m$ map. The following are equivalent.*

1. *f is surjective,*
2. *f is a quotient by some congruence K,*
3. *f is $\mathbb{P}$ regular epi,*
4. *f $\mathbb{P}^j_m$-regular epi,*
5. *f is $\mathbb{P}^j_m$-extremal epi.*

Proof. The basic facts about pocongruences show $(1 \iff 2)$. The preceding lemma shows $(1 \iff 3)$. $(4 \implies 5)$ holds in any category.

To show $(2 \implies 4)$, assume $f : A \to A/K$. Since $K \subseteq A \times A$, we have projection maps $\pi_1, \pi_2 : K \to A$. f is plainly the coequalizer of π_1, π_2.

($5 \implies 2$) holds because we may factor any map f through its associated pocogruence. If f is extremal epi, then f must coincide with the quotient by its associated pocongruence.

$\square$

Corollary 6.3.3. *Let j and m be subset systems which preserve regular epis. Let $m \geq \omega$. The forgetful functors $U_{\mathbb{P}} : \mathbb{P}^j_m \to \mathbb{P}$ and $U_{\mathbf{Set}} : \mathbb{P}^j_m \to \mathbf{Set}$ preserve and reflect regular epis.*

Lemma 6.3.4. *The forgetful functor $U_{\mathbb{DP}^j_m} : \mathbb{DP}^j_m \to \mathbf{Set}$ reflects kernel pairs.*

Proof. One verifies that the kernel pair of a map $f : A \to B$ (calculated in $\mathbb{DP}^j_m$) consists of the projections (from $A \times A$) restricted to the set

$$\{(x, y) \in A \times A : f(x) = f(y)\}.$$

$\square$

Applying Theorem 3.3.5, one obtains the following.

Corollary 6.3.5. *Let j and m be subset systems which preserve regular epis. Let $m \geq \omega$. The forgetful functor $U_{\mathbf{Set}} : \mathbb{DP}^j_m \to \mathbf{Set}$ is monadic.*

6.4 Coequalizers in $\mathbb{DP}^j_m$

Theorem 6.4.1. *For any pair $g, h : A \to B \in \mathbb{DP}^j_m$, the coequalizer* $\operatorname{coeq}(g, h)$ *exists.*

Proof. The construction of the coequalizer of maps g and h is illustrated by the following diagram. The maps on the diagram will be defined below.

$$\begin{tikzcd}
A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & B \arrow[r, "\alpha"] \arrow[dddr, "f"] & B/\preceq \arrow[r, "\eta(B/\preceq)"] \arrow[drr, "j"] \arrow[ddd, "\overline{f}"] & D(B/\preceq) \arrow[rr, "k"] \arrow[ddd, "D(\overline{f})"] & & E(B/\preceq) \arrow[ddd, "E(\overline{f})"] \\
 & & & & E_0 \arrow[ur] \arrow[d, "*"] & \\
 & & & & C \arrow[dr, "q"] & \\
 & & C \arrow[r, "\eta C"'] \arrow[urr, "m"] & D(C) \arrow[rr, "\ell"] & & E(C)
\end{tikzcd}$$

Let $\alpha : B \to B/\preceq$ be the map given by Construction 6.2.6. Let $f : B \to C$ be any $\mathbb{DP}^j_m$ map such that $fg = fh$. The existence and uniqueness of $\overline{f}$ is given by 6.2.6.

To construct the leftmost square, we apply the free completely distributive complete lattice functor $D = \mathfrak{UD}$; η is the natural transformation which injects a poset H into DH.

To construct the right-outside square, we note that (∞, ∞)-pocongruences are closed under intersection. So there is a smallest (in the sense that it makes the fewest possible identifications) (∞, ∞)-quotient

$$k : D(B/\preceq) \to E(B/\preceq)$$

such that $k(\eta(B/\preceq))\alpha$ preserves all (j, m)-optimum bounds. Similarly, define

$$\ell : D(C) \to E(C)$$

to be the smallest quotient such that $\ell(\eta C)f$ preserves all (j, m)-bounds. Since

$$\ell D(\overline{f})(\eta(B/\preceq))\alpha = \ell(\eta C)\overline{f}\alpha = \ell(\eta C)f,$$

preserves all (j, m)-bounds we have the induced map $E(\overline{f})$.

Define E_0 to be the smallest $\mathbb{DP}^j_m$-subobject of $E(B/\preceq)$ through which $k(\eta(B/\preceq))$ factors. (By Proposition 5.3.15, the intersection defining E_0 exists.) The maps $j : B/\preceq \to E_0$ and $E_0 \to E(B/\preceq)$ are obtained by factoring $(\eta(B/\preceq))\alpha)$ through E_0.

By construction, the map $\ell(\eta C) : C \to E(C)$ has the universal property that any map (j, m)-optimum bound preserving $\phi : C \to B$, with $B \in \mathrm{Obj}(\mathbb{DP}^j_m)$ factors as $\phi = t\ell(\eta C)$ for a uniquely determined map t. Since $C \in \mathrm{Obj}(\mathbb{DP}^j_m)$, there is such a ϕ which is $\mathbb{P}$-extremal mono. Thus, $\ell(\eta C)$ is $\mathbb{P}$-extremal mono. Therefore, the $\mathbb{P}$-(epi, extremal mono) factorization of $\ell(\eta C)$ produces the factorization $\ell(\eta C) = qm$. By construction, m is both $\mathbb{P}$-epi and $\mathbb{P}$-extremal mono; so m is a $\mathbb{P}$-isomorphism. Moreover, $q : C \to E(C)$ is necessarily the smallest $\mathbb{DP}^j_m$-subobject of $E(C)$ through which $\ell(\eta C)$ factors.

We claim that $\mathrm{coeq}(g, h) = E_0$ via $j : B \to E_0$. It suffices to show the existence of a unique compatible map $* : E_0 \to C$. The commutativity of the largest rectangle in the diagram above implies that $k(\eta(B/\preceq))$ factors uniquely through the $\mathbb{DP}_m^j$-object $E(\overline{f})^{-1}(C) \subseteq E(B)$. Because E_0 is the smallest $\mathbb{DP}_m^j$-object through which $k(\eta(B/\preceq))$ factors, $E_0 \subseteq E(\overline{f})^{-1}(C)$. This insures the existence and uniqueness of $*$. $\square$

CHAPTER 7
(j,m)-SPACES

This chapter studies (j, m)-spaces obeying a T_0-style separation axiom. Section 7.1 defines spaces and describes a functorial Galois connection, which specializes to Galois connections between $\mathbb{DP}^j_m$ and $\mathbb{S}^j_m$. Section 7.2 develops a convenient description of epimorphisms in $\mathbb{S}^j_m$, which generalizes a known characterization of epimorphisms of T_0-spaces. Section 7.3 gives constructions of limits, similar to those for topological spaces. Section 7.4 describes quotient maps, and characterizes extremal and regular epis as quotient maps.

The last two sections are related to the problem of finding reflections in $\mathbb{S}^j_m$; Section 7.5 gives the flat spectrum (co)reflection on spatial objects – which is equivalent to a reflection on $\mathbb{S}^j_m$. This reflection on spaces is a generalization of the T_1-reflection of topological spaces.

Last, but not least, Section 7.6 partially describes the epicomplete $\mathbb{S}^j_m$- objects; the description is complete for T_0-spaces. Epicomplete T_0-spaces are chains with the specialization order. Products of epicomplete T_0-spaces are not epicomplete, so there is no functorial epicompletion in the category of T_0-spaces.

As mentioned in the introductory chapter, the research leading to this dissertation began in an attempt to find reflections and coreflections in categories of generalized topological spaces. After proving the results of this chapter, and reading Meseguer [25], the author realized that additional assumptions where required on subset systems to insure that subspaces could be reasonably defined. This realization prompted much of the thought summarized in Chapter 6 – in particular Section 6.3; the author wanted to find when the theory of this chapter was valid. These considerations, and construction of free $\mathbb{DP}^j_m$-objects, became the main focus of the dissertation. However, this state of affairs leaves many questions concerning reflections and coreflections in $\mathbb{S}^j_m$ untouched.

This chapter assumes slightly more background then the rest of the text. Closure operators are used without comment. If c is a closure operator,

$$\mathrm{fix}(c) := \{x : c(x) = x\}.$$

7.1 Spatial/Sober Functorial Galois Connection

Definition and Remarks 7.1.1. Let $APos$ denote the category of whose objects A consist of

(ob1) an underlying partially ordered set, denoted A,

(ob2) a family of designated subsets $\mathfrak{J}A$, such that $\bigvee S$ exists for all $S \in \mathfrak{J}A$,

(ob3) a family of designated subsets $\mathcal{M}A$, such that $\bigwedge S$ exists for all $S \in \mathcal{M}A$.

Items (ob2) and (ob3) of the data defining a member of Obj($APos$) will be referred to as, *the signature of* A. Note that a given poset A may have several possible signatures. A $APos$-map $\phi : A \to B$ is a function $A \to B$ such that:

(map0) ϕ is monotone, i.e., $a \leq b \in A \Rightarrow \phi(a) \leq \phi(b)$.

(map1) $\phi[\mathfrak{J}A] := \{\phi(S) : S \in \mathfrak{J}A\} \subseteq \mathfrak{J}B$, and $\forall S \in \mathfrak{J}A$, $\phi(\bigvee S) = \bigvee \phi(S)$;

(map2) $\phi[\mathcal{M}A] \subseteq \mathcal{M}B$, and $\forall S \in \mathcal{M}A$, $\phi(\bigwedge S) = \bigwedge \phi(S)$.

The category $A\mathcal{S}$ has objects $(X, \mathfrak{O}(X), \Sigma(X), \Delta(X))$ where X is a set, $\mathfrak{O}(X)$ is a family of subsets of X, and $\Sigma(X), \Delta(X)$ are families of subsets of $\mathfrak{O}(X)$, such that $(\mathfrak{O}(X), \Sigma(X), \Delta(X))$ is a $APos$-object, in which the optimum bound operations – meet and join – are the set theoretic operations of intersection and union. $A\mathcal{S}$ maps are functions $f : X \to Y$ such that $f^{-1} : \mathfrak{O}(Y) \to \mathfrak{O}(X)$ is a $APos$-map.

Definition and Remarks 7.1.2. We describe the functorial Galois connection between $A\mathcal{S}$ and $APos$.

1. $\mathfrak{O} : A\mathcal{S} \to APos$ is the contravariant functor which sends a space X to the $APos$-object $(\mathfrak{O}(X), \Sigma(X), \Delta(X))$, and a map f, to the associated inverse image map, which sends $U \in \mathfrak{O}(Y)$ to $f^{-1}(U) \in \mathfrak{O}(X)$.
2. A prime filter on $A \in \mathrm{Obj}(APos)$ is a set $P \subseteq A$ such that

   (pFil1) P is *increasing*; that is, if $a \in P$ and $a \leq b$ then $b \in P$.

   (pFil2) P is $\mathcal{J}$-*inaccessible*; that is, if $S \in \mathcal{J}A$ and $\bigvee S \in P$, then $S \bigcap P$ is non-empty.

   (pFil3) P is $\mathcal{M}$-*closed*; that is, if $S \in \mathcal{M}A$ and $S \subseteq P$, then $\bigwedge S \in P$.

For $A \in \mathrm{Obj}(APos)$, there is a natural bijection between prime filters, prime ideals (decreasing, $\mathcal{J}$-closed and $\mathcal{M}$-inaccessible subsets of A) and characters $A \to 2 \in A\mathcal{S}$: if $x : A \to 2$ is a map, then $x^{-1}(0)$ is a prime ideal and $x^{-1}(1)$ is a prime filter. x can be recovered either from $x^{-1}(0)$ or $x^{-1}(1)$. For $a \in A$, let $\mathrm{coz}(a)$ denote the collection of all prime filters containing a.

1. Let ΨA denote the collection of all prime filters of A. ΨA is a $A\mathcal{S}$-object if we define

   $\mathfrak{O}(\Psi(A)) = coz[A] := \{coz(a) : a \in A\}$,

   $\Sigma(A) := coz[[\mathcal{J}A]] := \{\{\mathrm{coz}(a) : a \in S\} : S \in \mathcal{J}A\}$, and

   $\Delta(A) := coz[[\mathcal{M}A]]$.

   By definition of maps in $A\mathcal{S}$, Ψ extends to a contravariant functor.

2. In Banaschewski and Bruns [3] it is verified that Ψ and $\mathfrak{O}$ are adjoint on the right, where the unit natural transformations have components

   $$\eta_A(a) = \mathrm{coz}(a)$$

   and

   $$\epsilon_X(x) = \{U \in \mathfrak{O}(X) : x \in U\}$$

In verifying the adjunction, one shows that

$$\operatorname{coz}\Big(\bigvee S\Big) = \bigcup\Big\{\operatorname{coz}(a) : a \in S\Big\}$$

for any $S \in \mathcal{J}(A)$, and

$$\operatorname{coz}\Big(\bigwedge S\Big) = \bigcap\Big\{\operatorname{coz}(a) : a \in S\Big\},$$

for any $S \in \mathcal{M}(A)$.

3. We also follow Banaschewski and Bruns [3] in defining *SpAPos* to be the full subcategory of *APos* containing all A such that η_A is an isomorphism, and $SoA\mathcal{S}$ to be the full subcategory of $A\mathcal{S}$ containing X all X such that ϵ_X is an isomorphism. These categories are said to consist of *spatial posets* and *sober spaces* (respectively). Banaschewski and Bruns [3] show that *SpAPos* is onto-reflective in *APos* and that $SoA\mathcal{S}$ is reflective in $A\mathcal{S}$.

In order to apply the results of this section, we identify $\mathbb{P}^j_m$ with the full subcategory of *APos*, containing all (j, m)-complete objects with appropriate signature, i.e.,

$$\mathrm{Obj}(\mathbb{P}^j_m) = \{A \in APos : \mathcal{J}A = j(A),\, \mathcal{M}A = m(A)\}.$$

Rather than $A\mathcal{S}$, we discuss subcategories that roughly correspond to the $\mathbb{P}^j_m$ categories. We use the following notations: given $X \in \mathrm{Obj}(A\mathcal{S})$ and $x \in X$, $\epsilon_X(x)$ denotes the class of all open sets (members of $\mathfrak{O}$) that contain x. We refer to members of $\mathrm{Obj}(A\mathcal{S})$ as *spaces.*

We assume all spaces satisfy a separation axiom, which generalizes the usual T_0 axiom. Namely, we require:

(sep) $\forall x, y \in X, \epsilon(x) = \epsilon(y) \Rightarrow x = y$

As in general topology, whenever the axiom (sep) holds for X, the *soberification* map ϵ_X is injective.

Define $\mathcal{S}^j_m$ by

$$\mathcal{S}^j_m = \{X \in A\mathcal{S} \,:\, \mathfrak{O}(X) \in \mathbb{P}^j_m, \text{(sep) holds for X}\}.$$

Remark 7.1.3. *To strengthen the analogy between topological spaces and $\mathcal{S}^j_m$-objects, we assume that the subset systems j and m preserve regular epimorphisms.*

The functorial Galois connection of Banaschewski and Bruns [3] restricts to a duality between $So\mathcal{S}^j_m$ and $Sp\mathbb{P}^j_m$. Because of (sep),

$$\epsilon : id_{\mathcal{S}^j_m} \to \mathfrak{O}\Psi$$

is a monoreflection on $\mathcal{S}^j_m$.

7.2 The Skula Topology and Extremal Monos

Assume j and m preserve regular epimorphisms. If $S \subseteq X \in \mathrm{Obj}(\mathcal{S}^j_m)$, we may put a $\mathcal{S}^j_m$-structure on S by defining

$$\mathfrak{O}(S) = \{U \cap S : U \in \mathfrak{O}(X)\}.$$

By analogy with general topology, we call S with this $\mathcal{S}^j_m$ a (j, m)-subspace. Because the poset map dual to the inclusion $S \subseteq X$ is onto, and j and m are subset systems that preserve regular epis, $\mathfrak{O}(S)$ is (j, m)-complete.

Prompted by the treatment of similar problems in T_0 and sober topological spaces, we define a topology on (underlying sets) of objects from $\mathcal{S}^j_m$; see Skula [30]. We define the *Skula topology* on $X \in \mathrm{Obj}(\mathcal{S}^j_m)$ to be the topology with base

$$\mathfrak{O}(X) \cup \{X \setminus U : U \in \mathfrak{O}(X)\}.$$

It may seem somewhat confusing to put an actual topology on a generalized topological space, but it is also useful. Corollaries 7.2.2 and 7.2.3 identify epimorphisms and extremal monomorphisms using the Skula topology. Without it, the results become more cumbersome to state and less intuitive.

Lemma 7.2.1. *The Skula-closure of $S \subseteq X$, denoted $b(S)$, is given by the formula*

$$b(S) = \{x \in X : \forall G_1, G_2 \in \mathfrak{O}(X), G_1 \cap S = G_2 \cap S \Rightarrow x \notin G_1 \Delta G_2\}$$

Proof. Let $b'(S)$ denote the right hand side of the given formula. The operator b' is order preserving, and increasing. If $S \subseteq U$, we have a trace map from $\mathfrak{O}(U) \to \mathfrak{O}(S)$, given by $tr(A) = A \cap S$. Note that $b'(S)$ is the largest subspace of X so that the trace map is injective. Thus, b' is also idempotent, and, in fact, a closure operator. It now suffices to show $\mathrm{fix}(b') = \mathrm{fix}(b)$.

We first show $\mathrm{fix}(b') \subseteq \mathrm{fix}(b)$. Suppose $b'(A) = A$, and let U be the complement of A and $x \in U$. We will give a Skula-open neighboorhood of x contained in U. If $x \notin b'(A)$, then we have $G_1, G_2 \in \mathfrak{O}(X)$ with $x \in G_1 \setminus G_2$ and $G_1 \cap A = G_2 \cap A$. Evidently, $G_1 \cap (X \setminus G_2)$ is a Skula neighboorhood of x contained in U.

For $\mathrm{fix}(b) \subseteq \mathrm{fix}(b')$, it suffices to show $S \cap (X \setminus T)$ is b' closed, whenever $S, T \in \mathfrak{O}(X)$. Let $x \notin b'(S \cap (X \setminus T))$, with $S, T \in \mathfrak{O}(X)$. Then we have $G_1, G_2 \in \mathfrak{O}(X)$ with

$$G_1 \cap S \cap (X \setminus T) = G_2 \cap S \cap (X \setminus T)$$

and $x \in G_1 \setminus G_2$. It follows that $x \notin S \cap (X \setminus T)$. □

Corollary 7.2.2. *Let j and m be subset systems which preserve regular epis. A map $f : X \to Y \in \mathcal{S}^j_m$ is epi if and only if $b(f(X)) = Y$.*

We say $f : X \to Y$ is a (j,m)-*subspace embedding*, if f corestricted to $f(X)$ – considered as a (j,m)-subspace of Y – is an $\mathcal{S}^j_m$ isomorphism.

Corollary 7.2.3. *Let j and m be subset systems which preserve regular epis. A map $f : X \to Y$ is extremal mono if and only if f is a (j,m)-subspace embedding and $b(f(X)) = f(X)$.*

Proof. Suppose f is extremal mono. We may factor $f = X \to b(f(X)) \to Y$; the first factor is epi, *ergo* an isomorphism. Conversely, consider the map $b(f(X)) \subseteq Y$. If this factors as gs, where s is epi with codomain Z, we can factor g as $Z \to b(g(Z)) \subseteq Y$. The composition $b(f(X)) \to Z \to b(f(X))$ is an epi containment of subspaces of Y. By 7.2.2, $b(f(X)) = b(g(Z))$, so s is an isomorphism. □

Corollary 7.2.4. *Let j and m be subset systems which preserve regular epis. Suppose $X \in \mathrm{Obj}(So\mathbb{S}^j_m)$ is sober. Then $S \subseteq X$ is sober if and only if $b(S) = S$.*

Proof. Since $So\mathbb{S}^j_m$ is monoreflective in $\mathbb{S}^j_m$ and Skula-closed subobjects are extremal, if $b(A) = A$, then A is sober. For the converse, suppose A is sober. We have the epi containment $A \subseteq b(A)$ of sober spaces. Since the containment is epi, its dual is injective. Since it is a containment, the dual is surjective. We now apply the fact that bijective maps in $\mathbb{P}^j_m$ are isomorphisms. ☐

Lemma 7.2.5. *Let $f, g : A \to B \in \mathbb{DP}^j_m$ and*

$$c = \mathrm{coeq}_{\mathbb{DP}^j_m}(f, g) : B \to C.$$

The $Sp\mathbb{P}^j_m$-coequalizer of (f, g) is obtained by following c with the spatial reflection.

Theorem 7.2.6. *Let j and m be subset systems which preserve regular epis. Suppose $f : X \to Y \in So\mathbb{S}^j_m$ is a map of sober objects, with dual $\phi = Tf$. The following are equivalent:*

1. *f is extremal mono.*
2. *ϕ is extremal epi.*
3. *f is a subspace inclusion.*
4. *ϕ is a quotient map (in $\mathbb{P}^j_m$) such that the codomain is spatial.*
5. *ϕ is onto.*
6. *ϕ is regular epi.*
7. *f is regular mono.*

Proof. ($1 \Leftrightarrow 2$) and ($6 \Leftrightarrow 7$) hold because $So\mathbb{S}^j_m$ and $Sp\mathbb{P}^j_m$ are dual. ($2 \Leftrightarrow 3$) follows from 7.2.3 and 7.2.4. ($3 \Rightarrow 4$) holds because the $So\mathbb{S}^j_m$-structure on a subspace is defined by a $\mathbb{P}^j_m$-congruence. ($6 \Leftrightarrow 4 \Leftrightarrow 5$) holds by 6.3.2 and 7.2.5. ($7 \Rightarrow 1$) holds in any category; see Lemma 2.1.9. ☐

7.3 Computing Limits

We describe how to calculate limits in $\mathcal{S}^j_m$. To this end we prove a result analogous to the fact in general topology that any family of subsets generates a smallest topology.

Lemma 7.3.1. *Let X be a set. If $\mathfrak{F}$ is a collection of families of subsets of X, such that each $\tau \in \mathfrak{F}$ is closed under j-unions and m-intersections, then $\sigma := \bigcap \mathfrak{F}$ is also closed under j-unions and m-intersections.*

Proof. For any $\tau \in \mathfrak{F}$, the inclusion map from $\sigma \to \tau$ preserves subset inclusion. Thus, for any $\tau \in \mathfrak{F}$, $j(\sigma) \subseteq j(\tau)$ and $m(\sigma) \subseteq m(\tau)$, so if $S \in j(\sigma)$ (resp. $S \in m(\sigma)$) then $\bigcup S \in \tau$ (respectively, $\bigcap S \in \tau$). Hence $\sigma = \bigcap \mathfrak{F}$ is closed under unions of j-families and intersections of m-families. ☐

If α is a family of subsets of X, we consider $\mathfrak{F}$, the collection of all (j, m)-complete families of sets of X that contain α; $\bigcap \mathfrak{F}$ is called the *(j, m)-complete family generated by α.*

Definition and Remarks 7.3.2. Defining $\mathcal{S}^j_m$-products. Consider a set $\{X_i\} \subseteq \mathrm{Obj}(\mathcal{S}^j_m)$. The underlying set of X is the cartesian product ΠX_i, with canonical projections $\pi_i : X \to X_i$. $\mathfrak{O}(X)$ is the (j, m)-complete family generated by sets of the form $\pi_i^{-1}(U)$ where $U \subseteq \mathfrak{O}(X_i)$.

1. The object X thus defined satisfies (sep): if $x, y \in X$ and $x \neq y$, there exists i such that $\pi_i(x) \neq \pi_i(y)$. Since X_i satisfies (sep), there is $U \in \mathfrak{O}(X_i)$ with $|U \cap \{\pi_i x, \pi_i y\}| = 1$. Thus there is $V \in \mathfrak{O}(X)$ (namely $V = \pi_i^{-1}(U)$) with $|V \cap \{x, y\}| = 1$.

2. X is the categorical product of $\{X_i\}$: since the underlying set of X is the cartesian product of $\{X_i\}$, whenever we have an object $Y \in \mathrm{Obj}(\mathcal{S}^j_m)$ and functions $f_i : Y \to X_i$, there is a unique function $f : Y \to X$ so that for all i, $\pi_i f = f_i$. If each f_i is an $\mathcal{S}^j_m$-map, then because of our definition of $\mathfrak{O}(X)$, f is also an $\mathcal{S}^j_m$-map.

Definition and Remarks 7.3.3. $\mathcal{S}^j_m$-equalizers. Let j and m be subset systems which preserve regular epis. Let $f, g : X \to Y$ be a pair of maps in $\mathcal{S}^j_m$. The set $E := \{x \in X :$

$f(x) = g(x)\}$ has a natural $\mathcal{S}^j_m$-structure $\mathfrak{O}(E) = \{U \cap E : U \in \mathfrak{O}(X)\}$. The relation R on $\mathfrak{O}(X)$ given by

$$URV \iff U \cap E = V \cap E$$

is a $\mathbb{P}^j_m$-congruence because

$$\bigcap_i U_i \cap E = \Big(\bigcap_i U_i\Big) \cap E$$

and

$$\bigcup_i U_i \cap E = \Big(\bigcup_i U_i\Big) \cap E$$

for any set E and family of subsets (U_i). Since $\mathfrak{O}(E)$ is the quotient of $\mathfrak{O}(X)$ by a $\mathbb{P}^j_m$-congruence, and j and m preserve regular epis, $\mathfrak{O}(E)$ is (j, m)-complete. E satisfies (sep), because X does. If $h : Z \to X$ equalizes f, g, then the image of h must be contained in E. Since h is an $\mathcal{S}^j_m$-map, the inclusion of the image of h in E is also an $\mathcal{S}^j_m$-map.

7.4 Quotients, Extremal and Regular Epis

Definition and Remarks 7.4.1. Let $X \in \mathrm{Obj}(\mathcal{S}^j_m)$, $Eqv(X)$ denote the set of equivalence relations on X, and $\mathfrak{P}$ denote the set of subfamilies of $\mathfrak{O}(X)$. We consider equivalence relations to be sets of ordered pairs. If $R \in Eqv(X)$ and $x \in X$, we use the notation $R(x) = \{y \in X : (x, y) \in R\}$ for the equivalence class of x.

1. If $R \in Eqv(X)$, we say $U \in \mathfrak{O}(X)$ is *R-saturated* if $(x, y) \in R$ and $x \in U$ imply that $y \in U$; let R^* denote the family of all R saturated members of $\mathfrak{O}(X)$. If $R_1 \subseteq R_2$, then $R^*_2 \subseteq R^*_1$.

2. If $\sigma \in \mathfrak{P}$, we define an equivalence relation σ^* on X by $(x, y) \in \sigma^*$ if and only if for each $U \in \sigma$, we have $x \in U \iff y \in U$. Note that $\sigma_1 \subseteq \sigma_2 \Rightarrow \sigma^*_2 \subseteq \sigma^*_1$.

3. If $\sigma \in \mathfrak{P}$, then each member of σ is σ^*-saturated. In symbols, $\sigma \subseteq \sigma^{**}$.

4. Suppose $R \in Eqv(X)$. $U \in \mathfrak{O}(X)$ is R-saturated if and only if for all $(x, y) \in R$ $x \in U \iff y \in U$. Thus, $R \subseteq R^{**}$.

5. To summarize, $*$ gives a Galois connection between $Eqv(X)$ and $\mathfrak{P}$.

Given $R \in Eqv(X)$, we define an $\mathcal{S}^j_m$-structure on the set X/R of equivalence classes, by defining $\mathfrak{O}(X/R)$ to be the family $\{\{R(x) : x \in U\} : U \in R^*\}$. Since arbitrary unions and intersections of saturated sets are saturated and since the inclusion of R^* in $\mathfrak{O}(X)$ is order preserving, $\mathfrak{O}(X/R)$ is (j, m)-complete.

We omit the proofs of the following lemmas, which are straightforward verifications.

Lemma 7.4.2. *If $f : X \to Y \in \mathrm{Map}(\mathcal{S}^j_m)$, $R \in Eqv(X)$, and f is constant on each equivalence class $R(x)$, then f factors as $X \to X/R \to Y$, where $X \to X/R$ is the quotient map, and $X/R \to Y$ is defined by $R(x) \mapsto f(x)$. For all $U \in \mathfrak{O}(Y)$, $f^{-1}(U)$ is R-saturated and in $\mathfrak{O}(X)$.*

Lemma 7.4.3. *Let $R_1, R_2 \in Eqv(X)$ with quotient maps $q_i : X \to X/R_i$. There is a quotient map $q : X/R_1 \to X/R_2$ such that $q_2 = qq_1$ if and only if $R_1 \subseteq R_2$.*

Lemma 7.4.4. *For $R \in Eqv(X)$ the following are equivalent:*

1. *$R^{**} = R$,*
2. *there is $\sigma \in \mathfrak{P}$ such that $R = \sigma^*$,*
3. *X/R satisfies (sep).*

Proof. The equivalence of the first two conditions is trivial. If $R = \sigma^*$ and $R(x) \neq R(y)$, then there exists $U \in \sigma$ such that $|U \cap \{x, y\}| = 1$. So $(2 \Rightarrow 3)$.

$(3 \Rightarrow 1)$ Since $R \subseteq R^{**}$, we have a quotient map $q : X/R \to X/R^{**}$ as described, in the preceding lemma. It suffices to show that q is injective. If $R(x) \neq R(y)$ and X/R satisfies (sep), then there is $U \in R^*$ with $|U \cap \{x, y\}| = 1$. Thus, $R^{**}(x) \neq R^{**}(y)$. $\square$

Corollary 7.4.5. *If X is a T_0 topological space and $R \in Eqv(X)$, then X/R is T_0 if and only if $R = R^{**}$.*

Lemma 7.4.6. *If* $f : X \to Y \in \mathcal{S}^j_m$ *and* $R \in Eqv(X)$ *is defined by*

$$R = \{(x,y) : f(x) = f(y)\}$$

then $R^{**} = R$

Proof. We use Lemma 7.4.2. Since Y satisfies (sep), and for all $U \in \mathfrak{O}(Y)$, $f^{-1}(U)$ is R-saturated, X/R satisfies (sep): if $x, y \in Y$ and $x \neq y$, then we have $U \in \mathfrak{O}(Y)$ containing exactly one member of $\{x, y\}$. □

Definition and Remarks 7.4.7. Coequalizers in $\mathcal{S}^j_m$. If $f, g : X \to Y$, let $\sigma = \{U \in \mathfrak{O}(X) : f^{-1}(U) = g^{-1}(U)\}$.

The coequalizer of f, g is the quotient map $c : Y \to Y/\sigma^*$. First, one shows that $cf = cg$: if $c(f(x)) \neq c(g(x))$ there is $U \in \sigma$ which contains exactly one of $\{f(x), g(x)\}$. But $f^{-1}(U) = g^{-1}(U)$, since $U \in \sigma$; this contradicts the fact that x is in exactly one of $f^{-1}(U), g^{-1}(U)$. This contradiction shows that $cf = cg$.

Suppose $h : Y \to Z$ and $hf = hg$. We claim that $h = wc$ for a unique w. Note that h factors as $Y \to Y/R \to Z$, where $R = \{(x,y) : h(x) = h(y)\}$. By Lemma 7.4.3, it suffices to show that $\sigma^* \subseteq R$. If $h(x) \neq h(y)$, then we have $U \in \mathfrak{O}(Z)$, which contains exactly one member of $\{h(x), h(y)\}$. Since $hf = hg$, $h^{-1}(U) \in \sigma$, so $(x, y) \notin \sigma^*$.

We say $f : X \to Y$ is a *quotient map*, if there is an isomorphism $i : Y \to X/R$, and an equivalence relation $R = R^{**}$ such that $R(x) = if(x)$, for all x.

Proposition 7.4.8. *For* $f : X \to Y \in \mathcal{S}^j_m$*, the following are equivalent:*

1. *f is a quotient map,*

2. *f is regular epi,*

3. *f is extremal epi.*

Proof. (1 ⇒ 2). Suppose that f is a quotient map, with associated equivalence relation $R = R^{**}$ and family of R-saturated sets, R^*. Define a $\mathcal{S}^j_m$-object Z to have underlying set

$\bigcup\{C \times C : C \in X/R\}$ – a disjoint union – and define $\mathfrak{O}(Z) = \mathcal{P}(Z)$. $\mathfrak{O}(Z)$ is complete (hence (j, m)-complete) so $Z \in \mathrm{Obj}(\mathcal{S}^j_m)$; moreover any set map from $Z \to X$ is a $\mathcal{S}^j_m$-map. We now define two maps $g, h : Z \to X$ by $g(x, y) = x$ and $h(x, y) = y$. By construction, f is the coequalizer of g, h as computed in **Set**. Since f is also a $\mathcal{S}^j_m$-quotient map, f is also the coequalizer of g, h in $\mathcal{S}^j_m$.

($2 \Rightarrow 3$) holds in any category – [13, 17.11].

To show ($3 \Rightarrow 1$) suppose f is extremal epi. We may factor f into a quotient map q followed by an injection i, using Lemma 7.4.2. Since f is epi, then i is also epi; it now follows, by the extremal property of f that i is an isomorphism. So f is a quotient map. ☐

7.5 Flat Spectra

Let j and m be subset systems which preserve regular epis. We now examine a monocoreflective category of $Sp\mathbb{P}^j_m$. To define the category, we need to put a partial order on ΨA, for $A \in \mathbb{P}^j_m$. (Recall that ΨA is the set of functions $A \to 2$ which preserve order and (j, m)-optimum bounds.) We think of $x, y \in \Psi A$ as characters $x, y : A \to 2$. We define $x \leq y$ to mean $\forall a \in A. x(a) \leq y(a)$. Clearly, $x \leq y$ if and only if $\epsilon(x) = \{a \in A : x(a) = 1\} \subseteq \epsilon(y)$. The relation $\leq$ is anti-symmetric by virtue of the fact that our spaces satisfy (sep).

Now we define $\mathbb{F}^j_m$ to be the full subcategory of $S \in \mathbb{P}^j_m$ containing all A with the property that for all $x, y \in \Psi(A)$, $x \leq y$ implies $x = y$. To show $F\mathcal{S}^j_m$ monocoreflective, we will use the category theoretic dual of the following theorem.

Theorem 7.5.1. [13, 37.1] *Suppose $\mathcal{A}$ is a cowellpowered complete category, and $\mathcal{B}$ is a full, isomorphism closed subcategory. $\mathcal{B}$ is epireflective if and only if it is closed under products and extremal subobjects.*

Proposition 7.5.2. *$\mathbb{F}^j_m$ is monocoreflective in $Sp\mathbb{P}^j_m$.*

Proof. $Sp\mathbb{P}^j_m$ is cowellpowered and cocomplete, because it is wellpowered, complete and has a coseparator 2. [13, 23.14] We show that $\mathbb{F}^j_m$ is closed under coproducts. Suppose the coproduct $\coprod A_i$ has characters $x < y$. Since $x \neq y$, there is an index i such that $x\mu_i \neq y\mu_i$. But $x\mu_i, y\mu_i$ are characters of A_i. Thus, some A_i has characters $x < y$.

We show that $\mathbb{F}^j_m$ is closed under extremal epis. Suppose A has flat spectrum and $A \to A/K$ is a quotient. If A is non-trival, then it cannot have constant characters. Thus, if A has flat spectrum, all characters are onto. Now we use 6.3.2 to conclude that characters of A/K are precisely characters of A whose kernel contains K. It follows that A/K has flat spectrum. $\square$

Remark 7.5.3. The theorem would remain true if we redefined $\mathbb{F}^j_m$ to be a full subcategory of $\mathbb{P}^j_m$ containing flat spectrum objects. The rationale for not using this definition of $\mathbb{F}^j_m$ is that non-spatial flat spectrum objects behave rather differently than spatial objects. Without assuming A spatial, the spectrum gives incomplete information. For instance, there are A with no characters whatsoever! These objects trivially have flat spectrum. We wish to avoid this problem.

7.6 Epicomplete objects in $\mathbb{S}^j_m$

Assume throught this section that j and m preserve regular epimorphisms.

Definition 7.6.1. *An object A in any category is epicomplete if any map $f : A \to B$ that is both epi and mono is an isomorphism.*

Note: any monoreflection map is epi; see Herrlich and Strecker [13, Section 36]. Soberification is a monoreflection on $\mathbb{S}^j_m$. Therefore, any epicomplete object in $\mathbb{S}^j_m$ must be sober. We now examine which sober spaces are epicomplete.

We recall the partial order on characters $A \to 2$ that was used in the preceding subsection. For a sober space, the points $x \in X$ are in bijective correspondence to $\mathbb{S}^j_m$-maps $\mathfrak{O}(X) \to 2$. If X is a space, $x \leq y$ means $\epsilon(x) \subseteq \epsilon(y)$. Note that if $U \in \mathfrak{O}(X)$, $x \in U$ and $x \leq y$, then $y \in U$; in other words, each $U \in \mathfrak{O}(X)$ is a increasing subset of X. Also note that any continuous map preserves this order.

Lemma 7.6.2. *Suppose $X \in \mathrm{Obj}(\mathbb{S}^j_m)$ with $\omega \subseteq j \cap m$.*

1. *X is a chain if and only if $\mathfrak{O}(X)$ is a chain.*

2. *If X is not a chain, then there is a (j, m)-structure τ on X, such that $\tau \subset \mathfrak{O}(X)$ and (X, τ) satisfies (sep).*

Proof. 1. If X is a chain, then the family $\mathfrak{I}$ of all increasing subsets of X is a chain (ordered by inclusion). Since $\mathfrak{O}(X) \subseteq \mathfrak{I}$, $\mathfrak{O}(X)$ is a chain. Conversely, if $\mathfrak{O}(X)$ is a chain, then the set of prime ideals of $\mathfrak{O}(X)$ is a chain, so X is a chain.

2. Suppose $\mathfrak{O}(X)$ is not a chain. Then we have incomparable $U, V \in \mathfrak{O}(X)$. Let $\tau = \{W \in \mathfrak{O}(X) : W \subseteq U \text{ or } U \subseteq W\}$. One may easily verify that τ is closed under all unions and intersections which exist in $\mathfrak{O}(X)$; since $j(\tau) \subseteq j(\mathfrak{O}(X))$ and $m(\tau) \subseteq m(\mathfrak{O}(X))$, this shows that τ is a (j, m)-complete.

We now verify that (X, τ) satifies (sep). Suppose $x, y \in X$ are distinct points; without loss of generality, there is $G \in \mathfrak{O}(X)$ such that $x \in G$ and $y \notin G$. If $x \in U$ then $G \cap U \in \tau$ separates x and y. If $x \notin U$ and $y \notin U$, then $G \cup U$ separates x and y. Lastly, if $x \notin U$, and $y \in U$, then U separates x and y. ☐

Theorem 7.6.3. *Let $X \in \mathrm{Obj}(So\mathcal{S}^j_m)$, $\omega \subseteq j \cap m$. We have the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). If $j = \infty$, then (iv) $\Rightarrow$ (i).*

(i) X is epicomplete

(ii) If $f : X \to Y$ is $So\mathcal{S}^j_m$-monic, then $f(X)$, $\mathrm{b}f(X)$, and X are all homeomorphic.

(iii) There is no (j, m)-structure, τ on X, such that $\tau \subset \mathfrak{O}(X)$ and (X, τ) satisfies (sep).

(iv) X is a chain satisfying (iii).

Proof. (i$\Rightarrow$ ii). Suppose $f : X \to Y$ is monic. By corestriction of f we obtain a $\mathcal{S}^j_m$-map $X \to f(X)$. Composing with the inclusion map of $f(X)$ in $\mathrm{b}f(X)$ we obtain a monic epic $So\mathcal{S}^j_m$ map with domain X; (i) forces this map to be an isomorphism.

(ii $\Rightarrow$ iii). Suppose τ is a (j, m)-structure on X and $\tau \subseteq \mathfrak{O}(X)$. The identity map on X is a continuous bijection $(X, \mathfrak{O}(X)) \to (X, \tau)$. Following this map by $\epsilon_{(X,\tau)} : (X, \tau) \to$

$\Psi\tau$, we obtain a mono epi map $(X, \mathfrak{O}(X)) \to \Psi\tau$. (ii) implies that all maps involved are isomorphisms.

(iii $\Rightarrow$ iv). This follows from Lemma 7.6.2.

(iv $\Rightarrow$ i). When $j = \infty$, we have more information about which (j, m)-structures correspond to a given partially ordered set. Suppose $(X, \leq)$ is a poset and τ is a topology on X such that $x \leq y$ if and only if each open set containing x also contains y. The union of the family $\{U \in \tau : x \notin U\}$ is open and equals $\{y \in X : y \not\leq x\}$. Thus, τ must contain the weak topology generated by the sets $\{y \in X : y \not\leq x\}$ for $x \in X$.

Suppose X is a chain, and $f : X \to Y$ is an epic monomorphism. Since f is injective and preserves order, $f(X)$ is a chain order isomorphic to X. *Ergo*, the $f(X)$ has a topology finer than the weak topology induced by the partial ordering. Since the corestriction $f : X \to f(X)$ is continuous, the topology on X is finer than the topology on $f(X)$. Thus, X and $f(X)$ are homeomorphic. Since X is sober, $f(X)$ is sober, which implies $b(f(X)) = f(X)$. Since f is epi, $b(f(X)) = Y$, so X and Y are homeomorphic. $\square$

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BIOGRAPHICAL SKETCH

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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