Math 274 Homework One

Choose to work out five among the following six problems. Extra credit is available if you work out all six problems. Due: Tuesday, Jan 22, 2008

(1) Families of subsets.
   (a) Count the subsets of $[n]$ that contain at least one odd number.
   (b) Count the k-sets in $[n]$ having no two consecutive integers.
   (c) Count the lists of subsets $A_0, A_1, \ldots, A_n$ such that $A_0 \subset A_1 \subset \ldots \subset A_n$.
Count the lists such that $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n$

(2) Let A_n be the set of permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. Let B_n be the set of n-tuples $(b_1, b_2, \cdots, b_n)$ such that $1 \le b_i \le i$ for each $i \in [n]$. Construct a bijection from A_n to B_n. (hint: Use induction on n, employing a bijection from A_{n-1} to B_{n-1} to construct a bijection from A_n to B_n. Below we illustrate the process for $n = 3$.)

A_3	321	231	213	312	132	123
B_3	111	112	113	121	122	123

(3) Count the positive integer solutions to $\sum_{i=1}^{n} x_i \le k$?

(4) Use the expressions for powers of i in terms of binomial coefficients to evaluate $\sum_{i=1}^{n}(i^3 + 6i^2 + 12i + 8)$.

(5) Prove the following identities by counting a set in two ways. Then use the two identities to obtain simple formulas that evaluate $\sum_{i=1}^{n}\sum_{j=1}^{n} \min\{i, j\}$ and $\sum_{i=1}^{n}\sum_{j=1}^{n} \max\{i, j\}$.
   a) $\sum_{i=1}^{n}\sum_{j=1}^{n} \min\{i, j\} = \sum_{k=1}^{n} k^2$;
   b) $\sum_{k=1}^{n} k^2 = 2\binom{n+1}{3} + \binom{n+1}{2}$.
   (Hint: count 3-tuples with special properties.)

(6) Prove $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$ by induction and by counting two ways.