Math 274 Homework One solution

(1) Families of subsets.

(a) Count the subsets of $[n]$ *that contain at least one odd number.*

There are 2^n subsets of $[n]$, and if a subset does not contain an odd number, it consists of even numbers (there are $\lfloor n/2 \rfloor$ even numbers), so there are $2^{\lfloor n/2 \rfloor}$ bad subsets. Therefore, there are $2^n - 2^{\lfloor n/2 \rfloor}$ subsets that contain at least one odd number.

(b) Count the k*-sets in* $[n]$ *having no two consecutive integers.*

There are $n-1$ pairs of consecutive integers, so there are $(n-1)\binom{n-2}{k-2}$ bad k-sets. Therefore, the number of good ones is $\binom{n}{k} - (n-1)\binom{n-2}{k-2}$.

There is also a direct counting given by Andrew Levin: $\{x_1, x_2, \ldots, x_k\}$ is a a k-set of $[n]$ with $x_i < x_{i+1} + 1$ if and only if $\{x_1, x_2 - 1, x_3 - 2, \ldots, x_k - k - 1\}$ is a k-set of $[n-k+1]$. Thus the number of required k-sets is $\binom{n-k+1}{k}$.

(c) Count the lists of subsets $A_0, A_1, \ldots, A_n$ *such that* $A_0 \subset A_1 \subset \ldots \subset A_n$*. Count the lists such that* $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n$*.*

For the first part, there are $n!$ of them, since if you list the elements in the order of their appearance in the A_is, they form a permutation of $[n]$, and every permutation of $[n]$ can be naturally corresponds to such a sequence (let A_i contain the first i elements).

For the second part, note that if $i \in A_j$, then $i \in A_l$ for all $l \geq j$. Thus it depends on which set is the first one where i is. For each number $i \in [n]$, i can be placed in $n+2$ sets (remember empty set), thus the number of set sequences is $(n+2)^n$.

(2) *Let* A_n *be the set of permutations of* $[n]$*, where* $[n] = \{1, 2, \ldots, n\}$*. Let* B_n *be the set of* n*-tuples* $(b_1, b_2, \cdots, b_n)$ *such that* $1 \leq b_i \leq i$ *for each* $i \in [n]$*. Construct a bijection from* A_n *to* B_n*. (hint: Use induction on* n*, employing a bijection from* A_{n-1} *to* B_{n-1} *to construct a bijection from* A_n *to* B_n*. Below we illustrate the process for* $n = 3$*.)*

A_3	*321*	*231*	*213*	*312*	*132*	*123*
B_3	*111*	*112*	*113*	*121*	*122*	*123*

From $n-1$ to n, we place i at the end of a n-tuple of B_{n-1} if n is inverted to the i-th place of the permutation in A_{n-1} which corresponds to the n-tuple of B_{n-1}.

(3) *Count the positive integer solutions to* $\sum_{i=1}^{n} x_i \le k$*?*

From $\sum_{i=1}^{n} x_i \le k$, we get $\sum_{i=1}^{n} x_i + t = k$, where $t \ge 0$ is difference between RHS and LHS. Let $t' = t + 1 \ge 1$, we get $\sum_{i=1}^{n} x_i + t' = k + 1$, and the number of positive integer solutions to it is $\binom{k}{n}$.

(4) *Use the expressions for powers of i in terms of binomial coefficients to evaluate* $\sum_{i=1}^{n}(i^3 + 6i^2 + 12i + 8)$*.*

Note that $i^3 = 6\binom{i}{3} + 6\binom{i}{2} + \binom{i}{1}$ and $\sum_{k=0}^{n}\binom{k}{r} = \binom{n+1}{r+1}$,

we have

$$\begin{aligned}
\sum_{i=1}^{n}(i^3 + 6i^2 + 12i + 8) &= \sum_{i=1}^{n}(i+2)^3 = \sum_{i=1}^{n}(6\binom{i+2}{3} + 6\binom{i+2}{2} + \binom{i+2}{1}) \\
&= \sum_{j=3}^{n+2}(6\binom{j}{3} + 6\binom{j}{2} + \binom{j}{1}) \qquad \text{let } j = i+2 \\
&= \sum_{j=0}^{n+2}(6\binom{j}{3} + 6\binom{j}{2} + \binom{j}{1}) \qquad \text{the first few terms are just 0} \\
&= \sum_{j=0}^{n+2}(6\binom{j}{3}) + \sum_{j=0}^{n+2}(6\binom{j}{2}) + \sum_{j=0}^{n+2}\binom{j}{1} \\
&= 6\binom{n+3}{4} + 6\binom{n+3}{3} + \binom{n+3}{2}.
\end{aligned}$$

(5) *Prove the following identities by counting a set in two ways. Then use the two identities to obtain simple formulas that evaluate* $\sum_{i=1}^{n}\sum_{j=1}^{n}\min\{i,j\}$ *and* $\sum_{i=1}^{n}\sum_{j=1}^{n}\max\{i,j\}$*.*

a) $\sum_{i=1}^{n}\sum_{j=1}^{n}\min\{i,j\} = \sum_{k=1}^{n} k^2$*;*

b) $\sum_{k=1}^{n} k^2 = 2\binom{n+1}{3} + \binom{n+1}{2}$*.*

(Hint: count 3-tuples with special properties.)

For a), we look at the following table:

i/j	1	2	3	$\cdots$	n
1	1	1	1	$\cdots$	1
2	1	2	2	$\cdots$	2
3	1	2	3	$\cdots$	3
n	1	2	3	$\cdots$	n

LHS counts the sum of the numbers in the table. RHS counts it in a different way: we can decompose the numbers into $\sum_{k=1}^{n} k^2$ ones.

For b), consider the 3-tuples (x, y, z) such that $x, y, z \in [n]$ and $x \geq y, x \geq z$. For LHS, when $x = k$, both y and z are in $[k]$, so there are k^2 choices. For RHS, we consider the 3-tuples (x', y, z) with $x' = x + 1$. Now $x', y, z \in [n + 1]$ and $x' > y, x' > z$. When $y \neq z$, we have $\binom{n+1}{3}$ choices for x', y, z and each choice can form two required 3-tuples. When $y = z$, there are $\binom{n+1}{2}$ choices for $x', y = z$, and each choice gives one required 3-tuple.

Note that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\min\{i,j\}+\sum_{i=1}^{n}\sum_{j=1}^{n}\max\{i,j\} = \sum_{i=1}^{n}\sum_{j=1}^{n}(i+j) = \sum_{i=1}^{n}(n\cdot i+n(n+1)/2) = n^2(n+1),$$

thus $\sum_{i=1}^{n}\sum_{j=1}^{n}\max\{i,j\} = n^2(n+1) - \sum_{k=1}^{n} k^2$.

(6) *Prove* $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$ *by induction and by counting two ways.*

I just ignore the induction part.

For the counting by two ways, the RHS counts the $n+1$-words $a_1a_2\cdots a_{n+1}$ such that $a_i \leq i$ for all i, and for at least one j, $a_j \neq j$. For the LHS, we count the same objects, with n cases. For the k-th case, a_{k+1} is the number with largest index i such that $a_i \neq i$ and $a_j = j$ for all $j > i$.