Math 274 Homework Three–solution

(1) *Prove recursively that in each below, a_n is the n^{th} Catalan number.*
(a) Let a_n be the number of non-crossing pairings of $2n$ points on a circle.

Label the points as $1, 2, \ldots, 2n$. Then 1 is paired up with $2, 4, \ldots, 2(n-1)$. Consider the case when 1 is paired up with $2k$. (Note that 1 cannot pair up with an odd number) There are $a_{k-1}a_{n-k}$ non-crossing pairings with 1 pairs with $2(k-1)$. So $a_n = \sum_k a_{k-1}a_{n-k}$. Note that $a_0 = 1$.

(b) *let a_n be the number of configurations of pennies on a base row of n pennies, where pennies can be added so that each penny not in the base rests on two in the row immediately below it.*

Consider the case when the first gap is after kth penny (i.e., there is no penny resting on kth and $k+1$th pennies in the base). Then there are $k-1$ pennies on the second row which are all lying on the first k pennies in the first row before the gap, and there are $n-k$ pennies in the first row after the gap. Thus there $a_{k-1}a_{n-k}$ configurations for this case. So $a_n = \sum a_{k-1}a_{n-k}$. Note that $a_0 = 1$.

(2) *Let $b_{n,k}$ be the number of k-element subsets of $[n]$ containing no two consecutive integers. Obtain a recurrence relation in two indices for these numbers.*

There are two kinds of k-subsets of $[n]$: ones contain n and ones do not contain n. If a k-set contains n, then $n-1$ is not in it, so there are $b_{n-2,k-1}$ such k-subsets. There are $b_{n-1,k}$ k-sets not containing n. So $b_{n,k} = b_{n-2,k-1} + b_{n-1,k}$.

(3) *Let h_n equal the number of different ways in which the squares of a 1-by-n chessboard can be colored, using the colors red, white and blue so that no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then find a formula for h_n.*

Depend on the color of the nth position, we have two cases: if it is not colored red, then it has two possible colors and the remaining positions can be colored in h_{n-1} ways; if it is colored red, then the $n-1$-th position must be colored white or blue, and the other $n-2$ positions can be colored in h_{n-2} ways. So $h_n = 2h_{n-1} + 2h_{n-2}$ with $h_1 = 3$ and $h_2 = 6$.

(4) *Solve the following recurrence relation.*
$a_n = 5a_{n-1} - 6a_{n-2} + 2^n$.

The characteristic equation is $x^2 - 5x + 6 = 0$, and roots are 2 and 3. So the general solution is $a_n = A2^n + B3^n$ and a particular solution is $a_n = (Cn + D)n2^n$. With initial conditions, we are able to find the constants A, B, C and D.

(5) *Solve the following recurrence relation.*
$h_n = 5h_{n-1} - 6h_{n-2} - 4h_{n-3} + 8h_{n-4}$, $(n \geq 4)$, *with* $h_0 = 0, h_1 = 1, h_2 = 1, h_3 = 2$.

The characteristic equation is $x^4 - 5x^3 + 6x^2 + 4x - 8 = 0$, and the roots are 2 (with multiplicity 3) and -1 (with multiplicity 1). So the general solution is $h_n = A_1(-1)^n + B_1 2^n + B_2 n 2^n + B_3 n^2 2^n$. With initial conditions, we are able to find the coefficients (omit).