CHAPTER 1

Introduction

Logics come in many guises: both *semantic* and *syntactic*. In Classical Logic, the *truth* of a formula can be identified variously by: a line of $\top$s in a truth table; validity in Boolean algebras; even winning strategies in games. At the same time, a *proof* of the formula may be obtained in a (classical) Hilbert system built from axioms and a few simple rules, or a Gentzen sequent calculus built from simple axioms but many rules. These varied guises are useful: they exhibit different properties, reveal unsuspected connections, facilitate deeper algebraic investigation or are well suited for automated reasoning. There may be much more to a logic than a first (semantic or syntactic) glance might suggest.

The aim of this book is to show that logics with (semantic) roots in the real numbers $\mathbb{R}$, also have a natural (and useful!) characterization in proof theory. These logics often arise as the basis for systems dealing with vagueness; formalizing reasoning about natural language expression such as "tall" or "fast" in *Fuzzy Logic* [152]. For this reason, they are often known as, and are called here, *fuzzy logics*. But logics based on real numbers do in fact turn up in many areas in logic, mathematics, and computer science:

Example 1.1 (t-Norm Logics)**.** One way of building fuzziness into a logic is to make "design choices" at the outset. Take the real unit interval $[0, 1]$ as the set of truth values, 0 being falsity and 1 truth, and interpret connectives like "and" and "implies" by functions on $[0, 1]$ having reasonable properties. One good choice is to interpret "and" by *t-norms* (see Hájek [72]): binary functions that are commutative, associative, and increasing, and have 1 as a unit element. Each t-norm gives rise to a logic, for example:

- *Gödel Logic* **G** is based on the "minimum" t-norm $x * y = min(x, y)$: the only t-norm that interprets "A and A" as having the same truth value as "A". **G** has deep historical roots, being introduced by Dummett [41] in 1959 as the infinite-valued version of a family of finite-valued logics defined by Gödel [66] in the 1930s. It appears not only as an important fuzzy logic, but also as a key intermediate logic between Intuitionistic Logic and Classical Logic.

- *Łukasiewicz Logic* **Ł**, based on the "bounded sum" t-norm $x*y = max(0, x+y-1)$, is perhaps even more famous. It is the infinite-valued member of a family of many-valued logics introduced by Łukasiewicz [95, 97] in the 1920s. The so-called MV(many-valued)-algebras for **Ł** have been intensively investigated in their own right (even having a book devoted exclusively to them [34]).

- *Product Logic* $\mathbf{\Pi}$, based on the "product" t-norm $x \cdot y$, is a more recent addition to the fuzzy canon [76] (although see [68] for an early use of product *implication*).

Fuzzy logics may also be defined based on *classes* of t-norms; e.g. Hájek's Basic Logic $\mathbf{BL}$ based on continuous t-norms [72], and Godo and Esteva's Monoidal t-Norm Logic $\mathbf{MTL}$, the logic of left-continuous t-norms [46].

Example 1.2 (Expert Systems)**.** Logical reasoning based on the reals also turns up in expert systems; MYCIN [137] being a famous early example capable of reasoning under uncertainty. MYCIN diagnoses blood infections using *certainty factors* taken from the real interval $[-1, 1]$ and rules like:

IF the infection is primary-bacteremia
AND the site of the culture is one of the sterile sites
AND the suspected portal of entry is the gastrointestinal tract
THEN there is suggestive evidence (0.7) that infection is bacteroid.

To combine certainty factors MYCIN uses the function:

$$x * y \quad = \quad \begin{cases} x - y(1-x) & \text{if } min(x,y) \geq 0 \\ \dfrac{x+y}{1 - min(|x|,|y|)} & \text{if } min(x,y) < 0 < max(x,y) \\ x - y(1+x) & \text{if } max(x,y) \leq 0 \end{cases}$$

This complicated-looking function is an example of a *uninorm*: defined like t-norms but where the unit element can lie anywhere in $[0, 1]$ [151]. Unlike t-norms, uninorms allow for compensatory behaviour: new information can have either a negative (decreasing) or a positive (increasing) effect. Moreover, uninorms can be used in the same way as t-norms, to define fuzzy logics [102].

Example 1.3 (Resources)**.** In resource based logics *how often* a formula is used in a proof matters. In some, like Anderson and Belnap's relevance logics [3], they must be used at least once. In others, like Girard's Linear Logic [65], once exactly. One obvious way of modelling resources is to use *numbers*; for example:

- In Meyer and Slaney's *Abelian Logic* $\mathbf{A}$ [106] (also a comparative logic of Casari [24]), conjunction and implication are just ordinary addition and negation on $\mathbb{R}$. Truth is then associated with being greater than equal to 0.

- R-Mingle Logic $\mathbf{RM}$, a member of the Anderson and Belnap family [3], also has truth values in $\mathbb{R}$. However, in this case, having one copy of a formula is the same thing as having any number of copies. That is, conjunction is interpreted by the function:

$$x * y \quad = \quad \begin{cases} min(x, y) & \text{if } x \leq -y \\ max(x, y) & \text{otherwise} \end{cases}$$

so $x * x$, x "and" x, is just x.

Example 1.4 (Residuated Lattices)**.** Real numbers also make good candidates for constructing *algebras*. Consider the real line $\mathbb{R}$ equipped with the usual order $\leq$, addition $+$, and substraction $-$: this is an example of an ordered abelian group. Indeed, it is a particularly useful example, since an equation holds in this algebra if and only if it holds in *all* ordered abelian groups. Such facts can generalized to the framework of *residuated lattices* [80, 146, 89], which in the commutative case, are algebras:

$$\langle L, \wedge, \vee, \odot, \rightarrow, \mathrm{t} \rangle$$

where $\langle L, \wedge, \vee \rangle$ is a lattice, $\langle L, \odot, \mathrm{t} \rangle$ is a monoid, and $\rightarrow$ is the residuum of $\odot$, i.e. $x \odot y \leq z$ iff $x \leq y \rightarrow z$. Interesting candidates for commutative residuated lattices are obtained when L is $[0, 1]$ and $\wedge$ and $\vee$ are min and max, respectively. In this case we can ask which variety is *generated* by such algebras, and whether it can be (finitely) axiomatized.

The examples above give some idea of the kinds of logics and algebras that we are interested in. Our aim is to show that such logics have a natural syntactic or *proof-theoretic* characterization. By this, we mean that instead of saying when formulas are true by interpretations using algebras, we define algorithmic methods that tell us when they are provable. We consider two main approaches.

Hilbert systems (also known as Frege systems or axiomatizations) generate theorems from a stock of formulas (called axioms) using a small number of rules; often just one, the modus ponens rule: from A and $A \rightarrow B$, conclude B. Advantages of this approach are generality - we can think of as many weird and wonderful axioms as we want - and a close kinship with algebra. On the other hand, when it comes to actually reasoning and working with proofs, Hilbert systems prove extremely cumbersome. The problem is that to prove a theorem using rules like modus ponens, it is constantly necessary to guess which formulas should appear in applications of the rule.

Gentzen systems, introduced by Gentzen in the 1930s [59], are much better when it comes to reasoning about proofs. They gain flexibility by dealing with structures, called *sequents*, that look something like:

$$\Gamma \Rightarrow A$$

Here A is a formula and Γ is a structured collection of formulas: a set, multiset, or sequence. The sequent arrow $\Rightarrow$ is interpreted as "entails" or reading backwards "follows from". Intuitively, the sequent says that from the collection of premises Γ, we can deduce A. Now let us write Γ, A for the addition of a formula A to Γ: often we can interpret this "," as an "and". We can then write rules like:

$$\text{If } \Gamma \Rightarrow A \text{ and } \Gamma \Rightarrow B, \text{ then } \Gamma \Rightarrow A \wedge B$$

Sequent systems can be defined for a wide range of logics. In particular, we can switch between logics by adding and removing structural rules to obtain substructural logics [130]; Girard's Linear Logic [65] being one significant example. Fuzzy logics do not fit comfortably into this framework, however. To get Gentzen systems for these logics we need to treat sequents "in parallel": as so-called *hypersequents*:

$$S_1 \mid \ldots \mid S_n$$

where $S_1 \ldots S_n$ are sequents and the "|" is read as an "or". Since hypersequents *generalize* ordinary sequents, this framework covers logics defined using sequent calculi and also many others; obtained via rules that allow *interaction* between sequents.

Most important fuzzy logics can be defined in these frameworks. In particular, for Gentzen systems, fuzzy logics occur naturally as "hypersequent versions" of ordinary substructural logics. The key property for such systems is the existence of so-called "analytic" proofs: proofs where the formulas occurring are built from the same material (subformulas) as the formula proved. The existence of such proofs - established using the fundamental proof-theoretic technique of *cut-elimination* - has very nice consequences. In many cases we can immediately deduce decidability and complexity results, obtain interpolation and conservative extension properties, or use the calculus as the basis for automated reasoning methods. Most interesting of all, we can use Gentzen systems to tackle one of the main problems addressed in this book (and indeed a central topic in the fuzzy logics literature): *standard completeness*, showing that the semantic and syntactic approaches coincide.

Overview of the Book

The intended audience of this book includes readers unfamiliar with either fuzzy logics or proof theory (possibly both). For the former, we provide an accessible introduction to important fuzzy systems and results, semantic as well as syntactic. For the latter, we give a proof-theoretic presentation of fuzzy logics with significant applications to more traditional problems in the area. Below we give a brief overview of the contents:

- *Chapter 2* introduces the semantic building blocks of fuzzy logics: ordered sets of truth values equipped with functions to interpret connectives. We pay particular attention to developing the popular (and natural) t-norm approach, emphasizing the importance of the "fundamental" Łukasiewicz, Gödel, and Product t-norms. We also generalize the setting to cover residuated uninorms, and, with the help of a little universal algebra, define fuzzy logics in the framework of commutative residuated lattices.

- *Chapter 3* introduces Hilbert systems for fuzzy logics, built by extending a core set of axioms and rules with axioms reflecting key properties of the logics. A general proof is given that such logics are complete with respect to a corresponding subvariety of commutative residuated lattices. This result is then refined in several steps. First, we obtain completeness with respect to chains - linearly ordered members of the variety. Then, by extending logics with a special "density rule", completeness is obtained with respect to dense chains, and in many cases, algebras based on the real numbers.

- *Chapter 4* covers Gentzen systems. Sequents are introduced as a suitable framework for defining calculi for Classical Logic, Intuitionistic Logic, and many substructural logics. We discover, however, that when it comes to fuzzy logics, sequents are not enough. The main focus of the chapter is then a presentation of a large family of fuzzy logics as substructural logics in the framework of hypersequents. Soundness and completeness for sequent and hypersequent calculi (with cut) are established with respect to Hilbert systems.

- *Chapter 5* is the most technically demanding of the book, covering syntactic eliminations of rules from calculi. First, general conditions are given for structural rules that ensure cut-elimination for hypersequent calculi. Consequences of this include the subformula property and in certain cases decidability and independence results. We then treat the elimination of the density rule and show how this gives standard completeness for a large class of fuzzy logics.

- *Chapter 6* treats proof theory for the fundamental fuzzy logics: Łukasiewicz Logic, Gödel Logic, and Product Logic, and (aside from basic notions) may be read independently of Chapters 3-5. In the case of $\mathbf{G}$, we explore alternatives to the hypersequent approach of the previous section. For **Ł** and $\mathbf{\Pi}$, we provide both hypersequent and sequent calculi with non-standard rules and non-standard interpretations. Finally, we explore an extended framework, so-called relational hypersequents, in which the three logics are presented uniformly.

- *Chapter 7* explores complexity and efficiency issues. It is mostly concerned with the fundamental fuzzy logics and can therefore be read as a direct continuation of Chapter 6. First, proof systems developed in the previous chapter are used to obtain complexity results for the corresponding fuzzy logics. We then show how our systems can be extended to give a genuinely algorithmic presentation of the logics following the "logic programming style" goal-directed methodology.

- *Chapter 8* treats the addition of first-order quantifiers to fuzzy logics, providing algebraic, axiomatic, and Gentzen-style extensions.

- *Chapter 9* covers a variety of miscellaneous topics, slightly out of the scope of the main text. In particular, we consider the extension of fuzzy logics with modalities/truth-stressers and propositional quantifiers. We also consider related families of logics: non-commutative fuzzy logics, finite-valued logics, and comparative logics. Finally, we make some comments on the tricky case of Basic Logic and other open problems

Let us say finally what we leave out. There are already a number of fine texts on algebraic aspects of fuzzy and substructural logics [72, 34, 70, 130] and aggregation operators like t-norms [91]. Hence we follow the maxim here of including only what we really need; giving references to the literature at the end of each chapter. With regards to proof theory, we have made the conscious choice to treat just Hilbert and Gentzen systems. Of course many other proof-theoretic frameworks could be used: Natural Deduction, Tableaux, Display Logic, Calculus of Structures, to name just a few. These approaches are too similar to Gentzen systems to offer a really interesting alternative perspective, and - to our eyes at least - do not have the same clarity or convenience. Hypersequents are in a sense the minimal extension of sequents necessary to cover a wide spectrum of fuzzy logics. No doubt by extending the formalism further in the direction of Display Logic or Calculus of Structures further systems could be captured, but with an accompanying loss of clarity. One of the virtues of our presentation is that fuzzy logics occur naturally as substructural logics.