

Homotopy of Ends and Boundaries of CAT(0) Groups

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Abstract

For $n \geq 0$, we exhibit CAT(0) groups that are n -connected at infinity, and have boundary which is $(n - 1)$ -connected, but this boundary has non-trivial n^{th} -homotopy group. In particular, we construct 1-ended CAT(0) groups that are simply connected at infinity, but have a boundary with non-trivial fundamental group. Our base examples are 1-ended CAT(0) groups that have non-path connected boundaries. In particular, we show all parabolic semidirect products of the free group of rank 2 and \mathbb{Z} have a non-path connected boundary.

1 Introduction

A locally finite and connected complex Y is *simply connected at infinity* if given any compact set $C \subset Y$, there exists a compact $D \subset Y$ such that loops in $Y - D$ are homotopically trivial in $Y - C$. A finitely presented group G is simply connected at infinity if given some (equivalently any) finite connected complex X with $\pi_1(X)$ isomorphic to G , the universal cover of X is simply connected at infinity. This note centers around our examples, their implications and the natural questions that arise about possible examples in associated settings.

We present two base examples that induce the type of pathology we are interested in, but in entirely different ways.

The first example is a group that C. Croke and B. Kleiner [5] used to exhibit a CAT(0)group with more than one boundary. Croke and Kleiner recently observed [6] that the “right angled” boundary of this group is not path connected. As this group is 1-ended, any of its boundaries are connected.

The group in question is denoted H and has presentation

$$\langle a, b, c, d : [a, b] = [b, c] = [c, d] = 1 \rangle$$

and hence decomposition $\langle a, b \rangle *_{\langle b \rangle} \langle b, c \rangle *_{\langle c \rangle} \langle c, d \rangle$ (with all vertex groups isomorphic to $\mathbb{Z} \times \mathbb{Z}$). Completely analogous arguments show the groups $P = \langle t, x, y : [x, y] = 1, t^{-1}xt = y \rangle$ and $Q = \langle x, y : [x^2, y^2] = 1 \rangle$ have connected but non-path connected CAT(0) boundary. The group P is a “parabolic” semi-direct product of F_2 , the free group of rank 2, and \mathbb{Z} . Every parabolic semi-direct product of F_2 and \mathbb{Z} is isomorphic to a subgroup of finite index in P and so all such groups have a non-path connected CAT(0) boundary. The semi-direct products of F_2 and \mathbb{Z} are of three types, the parabolics, those such that some iterate of the automorphism is an inner automorphism and those that are “hyperbolic”. If an iterate of the automorphism is inner, then the only possible CAT(0) boundary for a resulting semi-direct product is the (path connected and non-locally connected) suspension of a Cantor set. In [18], K. Ruane shows the only possible CAT(0) boundary for a semi-direct product arising from a hyperbolic automorphism is the (path connected and locally path connected) Sierpinski carpet. The group Q is shown to be a semi-direct product of the free group of rank 3 and \mathbb{Z} . The groups P and Q are in a sense the smallest ones for which our techniques work.

The Cayley 2-complex K of the above presentation of H is a contractible 2-complex that is a union of planes any two of which have empty intersection or a line of intersection. If each square of K is given the geometry of $[0, 1] \times [0, 1]$, (the “right angled” geometry), then K becomes a CAT(0) space and H acts geometrically (discontinuously by isometries and cocompactly) on K . Our goals for this example are as follows. We first detail a proof of Croke-Kleiner’s observation that ∂K , the boundary of K , (with this geometry) is not path connected. As H is 1-ended, the direct product $H \times \mathbb{Z}$ is simply connected at infinity (see [12] or [20]). I.e. $\pi_1^\infty(H \times \mathbb{Z}) = 0$. A boundary of $H \times \mathbb{Z}$ is the boundary of $K \times \mathbb{R}$, which is simply the (unreduced) suspension of ∂K . (The *unreduced suspension* $S(X)$, of a Hausdorff space X is the quotient space of $X \times [0, 1]$, with $X \times \{0\}$ and $X \times \{1\}$ identified to (separate) points). As ∂K is not path connected, a simple Mayer-Vietoris argument shows that the rank of the first homology of the suspension of ∂K is 1 less than the cardinality of the set of path components of ∂K . In particular, $\pi_1(\partial(H \times \mathbb{Z}))$ is non-trivial. Using the Hurewicz and Mayer-Vietoris theorems repeatedly, we see that $H \times \mathbb{Z}^n$, for $n \geq 1$, is n -connected at infinity (see [7]), but has boundary with trivial homotopy in dimensions 0 through $n-1$ and non-trivial

n^{th} homotopy group. At the end of §3, it will be clear that a right angled Coxeter group version of H exists with the same pathology at infinity.

Our second example (labeled G) has various amalgamated product decompositions. This group acts geometrically on a 3-dimensional CAT(0) space with 2-dimensional boundary. We exhibit faithfully embedded suspensions of Hawaiian earrings joined at their common vertex in ∂G . The group G is easily seen to be simply connected at infinity, but ∂G (with right angled geometry) is not simply connected. The point of this example is to give a different construction than that for $H \times \mathbb{Z}$ of a group that is simply connected at infinity and has non-simply connected boundary. Models on G may be relevant to the questions in §6.

2 The boundary of H is not path connected

The proof here is not the same as that in [6]. The latter two authors of this paper only became aware of the argument in [6] after developing the proof presented in this section. Our argument works equally well to show the groups $P = \langle t, x, y : [x, y] = 1, t^{-1}xt = y \rangle$ (an HNN-extension of $\langle x, y \rangle \cong \mathbb{Z} \times \mathbb{Z}$) and $Q = \langle x, y : [x^2, y^2] = 1 \rangle$ have a non-path connected boundary.

Some background on the basics of CAT(0) geometry and connectivity at infinity for groups is necessary. Our basic reference for CAT(0)-geometry is [3] and for connectivity at infinity for groups [7]. A connected, locally finite complex Y is *semi-stable at ∞* if any two proper rays $r, s : [0, \infty) \rightarrow Y$ that converge to the same end of Y are properly homotopic. A finitely presented group G is *semistable at ∞* if for some (equivalently any) connected finite complex X with fundamental group G , the universal cover of X is semistable at ∞ . It is elementary to see that any two proper rays in \mathbb{R}^2 are properly homotopic, and any two proper rays that converge to the same end of \mathbb{R} are properly homotopic, so $\mathbb{Z} \times \mathbb{Z}$ and \mathbb{Z} are semistable at ∞ . The main result of [16] states that if a finitely presented group has a finite graph of groups decomposition with semistable at ∞ vertex groups and finitely generated edge groups, then that group is semistable at ∞ . An application of this result (see [17]) is that all 1-relator groups are semistable at ∞ . Hence the groups H , P and Q are semistable at ∞ . Our goal in this section is to define geodesic rays in CAT(0) universal covers for these groups that cannot be connected by a path in the boundary of this CAT(0) space (even though these rays are properly homotopic).

A $CAT(0)$ space is a geodesic metric space Y such that for any three points x, y, z of Y , the geodesic triangle (x, y, z) of Y is “at least as thin” as the comparison Euclidean triangle (x', y', z') . This means the distance in Y between any two points of (x, y, z) is less than or equal to the distance between the corresponding points of (x', y', z') . Consequentially, there are unique geodesics between points of Y , and so Y is contractible. A metric space Y is *proper* if for any $y \in Y$ and real $r > 0$, the ball of radius r at y is compact. The only $CAT(0)$ metric spaces considered in this paper are proper. For a $CAT(0)$ space Y and base point $y \in Y$, the set of geodesic rays $r : [0, \infty) \rightarrow Y$ based at y is given a topology and denoted ∂Y . The space ∂Y is independent of base point. When Y is proper, ∂Y compactifies Y so that the space $\partial Y \cup Y$ is compact metric (see [3]). The correct intuition is that if two geodesic rays in Y stay “close” for a long time, then they are close in ∂Y . For $A \subset Y$, the *limit set* of A in ∂Y is the intersection of ∂Y with the closure of A in $\partial Y \cup Y$. Denote it by $L(A)$. If $a \in L(A)$, there is a sequence $\{a_1, a_2, \dots\}$ in A such that $\lim_{i \rightarrow \infty} a_i = a$ where the limit is taken in $\partial Y \cup Y$. In particular, if A is a closed convex subset of Y , then for each $x \in L(A)$, there is a geodesic ray in A converging to x . A boundary of a $CAT(0)$ space Y is a *boundary for a finitely generated group G* if G acts *geometrically* (properly discontinuously and cocompactly by isometries) on Y .

The Cayley graph of the presentation $\langle a, b, c, d : [a, b] = [b, c] = [c, d] = 1 \rangle$ of H is a labeled graph with vertex set H , and a directed edge labeled t from the vertex u to the vertex v if $ut = v$ for some $t \in \{a, b, c, d\}$. Hence any edge path in this Cayley graph is labeled by elements in $\{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}\}$, where the label t^{-1} simply indicates traversing an edge labeled t in the direction opposite its orientation. The Cayley 2-complex of this presentation is obtained from the Cayley graph by attaching 2-cells at each vertex according to the relations. Let K be the Cayley 2-complex of our presentation of H with geometry such that each edge is isometric to $[0, 1]$ and each 2-cell to $[0, 1] \times [0, 1]$. With this geometry, K is a $CAT(0)$ complex. The remainder of this section is devoted to showing that the boundary of K is not path connected.

Consider the Cayley graph paths represented by infinite words

$$r = cdab(cb)^2cdab(cb)^6cdab\dots = \prod_{i=1}^{\infty} (cb)^{k_i} cdab$$

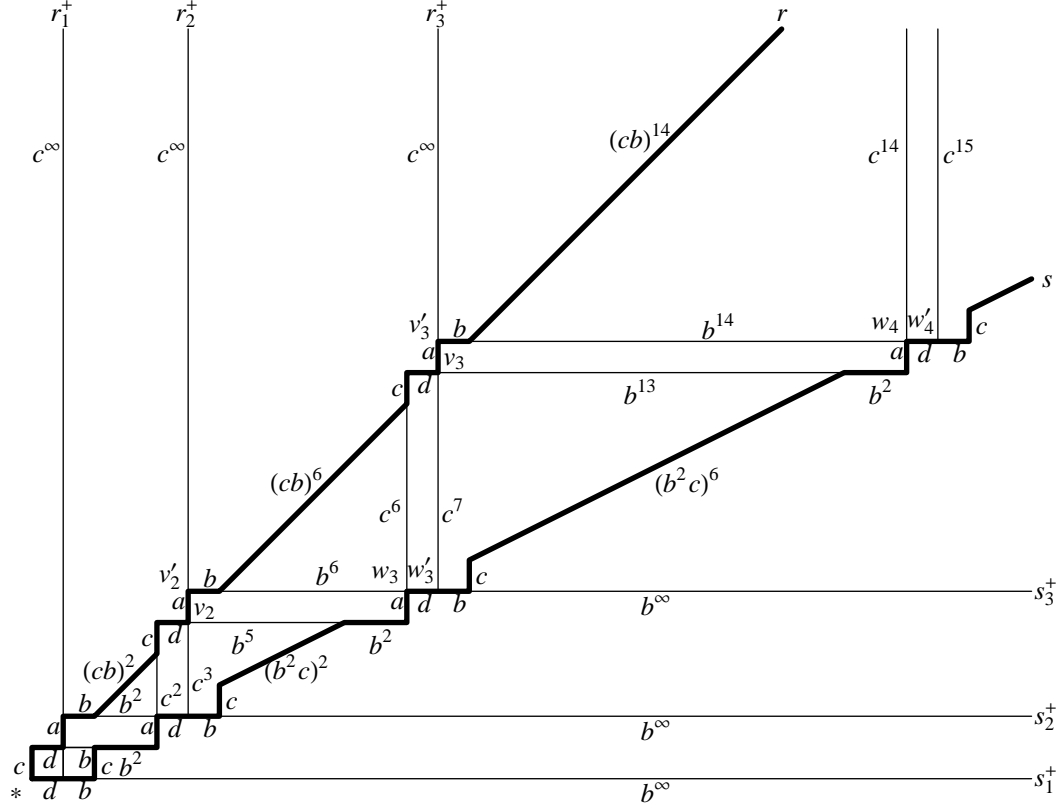


Figure 1: Schematic diagram of rays in $\langle a, b, c, d : [a, b] = [b, c] = [c, d] = 1 \rangle$.

and

$$s = dbcb^2adbc(b^2c)^2b^2adbc(b^2c)^6b^2a \dots = \prod_{i=1}^{\infty} dbc(b^2c)^{k_i}b^2a$$

where the exponents k_i are recursively defined by $k_0 = -1$ and, for all $i \geq 0$, $k_{i+1} = 2k_i + 2$ (see figure 1).

Let $v'_{-1} = b^{-1}$ and $w_0 = *$. For $n \geq 0$ define

$$v_n = \left(\prod_{i=1}^n (cb)^{k_i} cdab \right) (cb)^{k_{n+1}} cd$$

For $n \geq 1$ define

$$w_n = \left(\prod_{i=1}^n dbc(b^2c)^{k_i} b^2a \right)$$

For $n \geq 0$ define $v'_n = v_n a$ and $w'_n = w_n d$. Then $v_0 = cd$, $v'_0 = cda$, $v_1 = cdab(cb)^2cd$ and $w'_0 = d$, $w_1 = d b c b^2 a$. The $\langle b, c \rangle$ -cosets of K intersected by r (respectively s) are $v_i \langle b, c \rangle$ and $v'_{i-1} \langle b, c \rangle$ (respectively $w_i \langle b, c \rangle$ and $w'_i \langle b, c \rangle$) for all $i \geq 0$.

Claim 1 For all $n \geq 0$,

1. $v_n = v'_{n-1} b(cb)^{k_{n+1}} cd$
2. $w_{n+1} = w'_n bc(b^2c)^{k_{n+1}} b^2a$
3. $v_n = w'_n bc(b^2c)^{k_{n+1}} b^2 b^{-k_{n+2}-1}$ so $v_n = w'_n c^{k_{n+1}+1}$
4. $v'_{n-1} b(cb)^{k_{n+1}} c c^{-k_{n+1}-1} = w_n$ so $v'_{n-1} b^{k_{n+1}+1} = w_n$.

In particular, r and s pass through the same $\langle b, c \rangle$ -cosets of K , and v_n and w'_n (respectively v'_{n-1} and w_n) are in the same $\langle c \rangle$ -coset (respectively $\langle b \rangle$ -coset).

Proof: The first two identities follow directly from the definitions. In every case the third identity follows from the others by multiplying both sides of the fourth identity (on the right) by $dbc(b^2c)^{k_{n+1}} b^2 b^{-k_{n+2}-1}$ since

$$\begin{aligned} c^{-k_{n+1}-1} dbc(b^2c)^{k_{n+1}} b^2 b^{-k_{n+2}-1} &= dc^{-k_{n+1}-1} cc^{k_{n+1}} bb^{2k_{n+1}} b^2 b^{-k_{n+2}-1} \\ &= db^{2k_{n+1}+2-k_{n+2}} = d \end{aligned}$$

by the recursion relation. The fourth identity holds trivially for $n = 0$. If the third identity holds for a given n , then the fourth identity for $n + 1$ follows by multiplying both sides of the third (on the right) by $ab(cb)^{k_{n+2}} c c^{-k_{n+2}-1}$ since

$$b^{-k_{n+2}-1} ab(cb)^{k_{n+2}} c c^{-k_{n+2}-1} = ab^{-k_{n+2}-1} bb^{k_{n+2}} c^{k_{n+2}} cc^{-k_{n+2}-1} = a. \quad \square$$

The $\langle b, c \rangle$ -cosets traversed by r and s are $\mathcal{P}_i \equiv v_i \langle b, c \rangle$ for $i \geq 0$ and $\hat{\mathcal{P}}_i \equiv v'_i \langle b, c \rangle$ for $i \geq -1$. By claim 1, and the definitions of v'_n and w'_n , $w'_n \in \mathcal{P}_n$ and $w_n \in \hat{\mathcal{P}}_{n-1}$ for all $n \geq 0$. Since $v'_n = v_n a$, there is a unit wide

strip B_n connecting the line/coset $v_n\langle b \rangle$ of \mathcal{P}_n with the line $v'_n\langle b \rangle (= w_{n+1}\langle b \rangle)$ of $\hat{\mathcal{P}}_n$ for each $n \geq 0$. Each unit square of B_n has two opposite sides labeled b and the other two opposite sides labeled a . Since $w'_n = w_n d$, part 1) of claim 1, shows for $n \geq 0$ there is a unit wide strip C_n connecting the line $v'_{n-1}b(cb)^{k_{n+1}}\langle c \rangle (= w_n\langle c \rangle)$ of $\hat{\mathcal{P}}_{n-1}$ to the line $v_n\langle c \rangle (= w'_n\langle c \rangle)$ of \mathcal{P}_n . Each unit square of C_n has two opposite sides labeled c and the other two opposite sides labeled d . Portions of the strips B_n and C_n are drawn in figure 1.

Summarizing, the rays r and s have their images in the union of (and consecutively pass through) the sets $C_1, \mathcal{P}_1, B_1, \hat{\mathcal{P}}_1, C_2, \mathcal{P}_2, \dots$. The planes \mathcal{P}_n and $\hat{\mathcal{P}}_n$ contain parallel $\langle b \rangle$ -lines spanned by B_n , and the planes $\hat{\mathcal{P}}_n$ and \mathcal{P}_{n+1} contain parallel $\langle c \rangle$ -lines spanned by C_{n+1} . These sets are disjoint except for the those overlaps:

1. $C_n \cap \hat{\mathcal{P}}_{n-1} = v'_{n-1}b(cb)^{k_{n+1}}\langle c \rangle = w_n\langle c \rangle$,
2. $C_n \cap \mathcal{P}_n = v_n\langle c \rangle = w'_n\langle c \rangle$,
3. $B_n \cap \mathcal{P}_n = v_n\langle b \rangle$, and
4. $B_n \cap \hat{\mathcal{P}}_n = v'_n\langle b \rangle = w_{n+1}\langle b \rangle$.

From 2) and 3), $C_n \cap B_n = \{v_n\}$. From 1) and 4), $B_n \cap C_{n+1} = \{w_{n+1}\}$. Each $\langle b, c \rangle$ coset is a flat plane of K , its boundary is a circle corresponding to the slopes of rays from a point in the plane. Noting that our group can be represented as an amalgamated product

$$\langle a, b \rangle *_{\langle b \rangle} \langle b, c \rangle *_{\langle c \rangle} \langle c, d \rangle$$

we observe that for any vertex k of K , the line $k\langle b \rangle$ (respectively $k\langle c \rangle$) separates K . The group $\langle a, b, c \rangle = \langle a, c \rangle \times \langle b \rangle$ is the direct product of a 2-generated free group and an infinite cyclic group, its boundary being the suspension of a Cantor set with the $\langle b \rangle^{\pm\infty}$ points being the suspension points. For any vertex k of K , the set $k\langle a, b, c \rangle$ is isometric to the direct product $T \times \mathbb{R}$ of a valence 4 tree T and real line. The set $k\langle a, b, c \rangle - k\langle b \rangle$ has four components, but the points kc and kc^{-1} can be connected in $K - k\langle b \rangle$, by a path (beginning at kc^{-1}) with consecutive edge labels (d, c, c, d^{-1}) and so $K - k\langle b \rangle$ has only three components. Similarly $K - k\langle c \rangle$ has three components. It is necessary to understand the limit sets of these triples of components and, when k is a point of r or s , how the sets C_i, B_i, \mathcal{P}_i and $\hat{\mathcal{P}}_i$ are distributed among such components.

For k a vertex of K , let $Q(k\langle b\rangle, ka)$ be the component of $K - k\langle b\rangle$ containing ka , $Q(k\langle b\rangle, ka^{-1})$ the component containing ka^{-1} and $Q(k\langle b\rangle, kc)$ the component containing $kc^{\pm 1}$ and $kd^{\pm 1}$. Similarly define the components of $K - k\langle c\rangle$ to be $Q(k\langle c\rangle, kd)$, $Q(k\langle c\rangle, kd^{-1})$ and $Q(k\langle c\rangle, kb)$ (so this last component contains $kb^{\pm 1}$ and $ka^{\pm 1}$). For appropriate k , t and s , denote the closure of $Q(k\langle s\rangle, ks)$ in K by $\bar{Q}(k\langle t\rangle, ks)$ ($= Q(k\langle t\rangle, ks) \cup k\langle t\rangle$).

Claim 2 For $t = b$ and $s \in \{a, a^{-1}, c\}$ or $t = c$ and $s \in \{d, d^{-1}, b\}$

1. The set $\bar{Q}(k\langle t\rangle, ks)$ is convex in K .
2. $\partial K = L(\bar{Q}(k\langle b\rangle, ka)) \cup L(\bar{Q}(k\langle b\rangle, ka^{-1})) \cup L(\bar{Q}(k\langle b\rangle, kc))$
 $= L(\bar{Q}(k\langle c\rangle, kd)) \cup L(\bar{Q}(k\langle c\rangle, kd^{-1})) \cup L(\bar{Q}(k\langle c\rangle, kb))$
3. Any two sets in $\{L(\bar{Q}(k\langle b\rangle, ka)), L(\bar{Q}(k\langle b\rangle, ka^{-1})), L(\bar{Q}(k\langle b\rangle, kc))\}$ (respectively $\{L(\bar{Q}(k\langle c\rangle, kd)), L(\bar{Q}(k\langle c\rangle, kd^{-1})), L(\bar{Q}(k\langle c\rangle, kb))\}$) intersect in the 2-point set $L(k\langle b\rangle)$ (respectively $L(k\langle c\rangle)$). Furthermore, this 2-point set separates any two such sets in ∂K .

Proof: For part (1), the only way out of this set is through the geodesic line $k\langle t\rangle$. Since geodesics are unique in a CAT(0) space, convexity follows.

Part (1) implies that once a geodesic ray leaves one of these sets, it cannot return. Hence any geodesic ray, must eventually stay in one of the three sets. Part (2) follows.

If q is a ray in two limit sets of part (3), then by convexity, there are rays q_1 and q_2 in these sets respectively, that both stay a bounded distance from q . Hence they stay a bounded distance from $k\langle b\rangle$ and the first part of part (3) is proved. If $\alpha : [0, 1] \rightarrow \partial K$ connects points from two distinct such sets and α avoids $k\langle b\rangle$, then the inverse image under α of the three (closed) subsets of ∂K under consideration, give three disjoint closed sets whose union is $[0, 1]$ and two of these are non-empty. But $[0, 1]$ is connected, so (3) is proved. \square

By claim 1 $v_i\langle c\rangle = w'_i\langle c\rangle$ and $v'_i\langle b\rangle = w_{i+1}\langle b\rangle$. Define

1. $Q_i^+ = \bar{Q}(v_i\langle c\rangle, b) = \bar{Q}(w'_i\langle c\rangle, b)$.
2. $Q_i^- = \bar{Q}(v_i\langle c\rangle, d^{-1}) = \bar{Q}(w'_i\langle c\rangle, d^{-1})$
3. $\hat{Q}_i^+ = \bar{Q}(v'_i\langle b\rangle, c) = \bar{Q}(w_{i+1}\langle b\rangle, c)$
4. $\hat{Q}_i^- = \bar{Q}(v'_i\langle b\rangle, a^{-1}) = \bar{Q}(w_{i+1}\langle b\rangle, a^{-1})$.

Claim 3 For $k \geq i$,

1. $\hat{Q}_k^+ \cup Q_k^+ \cup \mathcal{P}_k \cup B_k \cup \hat{\mathcal{P}}_k \cup C_{k+1} \subset Q_i^+$,
2. $\hat{Q}_k^+ \cup Q_{k+1}^+ \cup \hat{\mathcal{P}}_k \cup C_{k+1} \cup \mathcal{P}_{k+1} \cup B_{k+1} \subset \hat{Q}_i^+$.
For $k \leq i$,
3. $\hat{Q}_{k-1}^- \cup Q_k^- \cup \mathcal{P}_{k-1} \cup B_{k-1} \cup \hat{\mathcal{P}}_{k-1} \cup C_k \subset Q_i^-$,
4. $\hat{Q}_k^- \cup Q_k^- \cup \hat{\mathcal{P}}_{k-1} \cup C_k \cup \mathcal{P}_k \cup B_k \subset \hat{Q}_i^-$. \square

The limit set of C_n (respectively B_n), $L(C_n)$ (respectively $L(B_n)$), is a 2-point set $\{c_n^{+\infty}, c_n^{-\infty}\}$ (respectively $\{b_n^{+\infty}, b_n^{-\infty}\}$) determined by the C_n (respectively B_n) geodesic rays from v_n with either all edges labeled c or all edges labeled c^{-1} (respectively b or b^{-1}).

Claim 4 For $n \geq 1$,

1. $\{c_n^{\pm\infty}\} = L(C_n) = L(w_n\langle c \rangle) = L(v'_{n-1}\langle c \rangle) = L(w'_n\langle c \rangle) = L(v_n\langle c \rangle)$,
2. $\{b_n^{\pm\infty}\} = L(B_n) = L(v_n\langle b \rangle) = L(v'_n\langle b \rangle) = L(w_{n+1}\langle b \rangle) = L(w'_n\langle b \rangle)$
3. $\{c_n^{\pm\infty}\} \subset L(\hat{\mathcal{P}}_{n-1}) \cap L(C_n) \cap L(\mathcal{P}_n)$
4. $\{b_n^{\pm\infty}\} \subset L(\mathcal{P}_n) \cap L(B_n) \cap L(\hat{\mathcal{P}}_n)$.

Proof: The equalities of the pairs of limits points in parts 1) and 2) follow directly from claim 1. Parts 3) and 4) follow from the definitions. \square

Recall, the rays r and s have their images in the union of (and consecutively intersect) the sets $C_n, \mathcal{P}_n, B_n, \hat{\mathcal{P}}_n, C_{n+1}, \mathcal{P}_{n+1}, \dots$. Let $L(r)$ and $L(s)$ be the limits sets of the image of r and s respectively. Later we show r and s track (diverging) geodesics and so $L(r)$ and $L(s)$ are distinct single element sets.

Claim 5 For $n \geq 1$,

1. $L(r) \cup L(s) \subset L(Q_n^+) \cap L(\hat{Q}_n^+)$.
2. $\cup_{i=n}^{\infty} \{c_i^{\pm\infty}, b_i^{\pm\infty}\} \subset L(\cup_{i=n}^{\infty} \mathcal{P}_i) \subset L(Q_n^+)$
3. $\{c_n^{\pm\infty}\} \cup (\cup_{i=1}^{n-1} \{c_i^{\pm\infty}, b_i^{\pm\infty}\}) \subset L(\hat{\mathcal{P}}_{n-1}) \cup L(\cup_{i=1}^{n-1} \mathcal{P}_i) \subset L(Q_n^-)$

4. The sets $\cup_{i=1}^{n-1}\{c_i^{\pm\infty}, b_i^{\pm\infty}\}$ and $(L(r) \cup L(s)) \cup (\cup_{i=n}^{\infty}\{c_{i+1}^{\pm\infty}, b_i^{\pm\infty}\})$ are separated in ∂K by $\{c_n^{\pm\infty}\}$.
5. $\cup_{i=n}^{\infty}\{c_{i+1}^{\pm\infty}, b_i^{\pm\infty}\} \subset L(\hat{\mathcal{P}}_n) \cup L(\cup_{i=n+1}^{\infty}\mathcal{P}_i) \subset L(\hat{Q}_n^+)$
6. $\cup_{i=1}^n\{c_i^{\pm\infty}, b_i^{\pm\infty}\} \subset L(\cup_{i=1}^n\mathcal{P}_i) \subset L(\hat{Q}_n^-)$
7. The sets $\{c_n^{\pm\infty}\} \cup (\cup_{i=1}^{n-1}\{c_i^{\pm\infty}, b_i^{\pm\infty}\})$ and $(L(r) \cup L(s)) \cup (\cup_{n+1}^{\infty}\{c_i^{\pm\infty}, b_i^{\pm\infty}\})$ are separated in ∂K , by $\{b_n^{\pm\infty}\}$.

Proof: The first inclusions of parts 2), 3), 5) and 6) follow from claim 4. The second inclusions follow from claim 3. Parts 4) and 7) follow from part 3) of claim 2, part 1), and respectively the two preceding parts of this claim. \square

Parts 4) and 7) of claim 5 imply our next result.

Claim 6 *If there is a path in $\partial(X)$ between either $c_i^{+\infty}$ or $c_i^{-\infty}$ and a point of $L(r)$, then it must pass through either $b_i^{+\infty}$ or $b_i^{-\infty}$. If there is a path in $\partial(X)$ between either $b_i^{+\infty}$ or $b_i^{-\infty}$ and a point of $L(r)$, then it must pass through $c_{i+1}^{+\infty}$ or $c_{i+1}^{-\infty}$. If there is a path from $c_1^{+\infty}$ to a point of $L(r)$, then it must successively pass through either $b_1^{+\infty}$ or $b_1^{-\infty}$, then through either $c_2^{+\infty}$ or $c_2^{-\infty}$, then through either $b_2^{+\infty}$ or $b_2^{-\infty}$, etc. \square*

In the remainder of this section we show r and s track geodesics that determine distinct points in ∂K , and there are geodesic rays r_n^{\pm} (respectively s_n^{\pm}) at $*$ that pass through v'_n (respectively w'_n) and have boundary points $c_n^{\pm\infty}$ (respectively $b_n^{\pm\infty}$). This implies the $b_n^{\pm\infty}$ (respectively $c_n^{\pm\infty}$) converge to the limit of r (respectively s) in ∂K .

Claim 7 *If $t \in \{v'_i, w'_i\}$ and $i \geq 0$, then t is the closest point of $t\langle b, c \rangle$ to $*$ (and the first point of r (if $t = v'_i$) and s (if $t = w'_i$) in $t\langle b, c \rangle$).*

Proof: Figure 1 is an appropriate diagram for this claim. Let x be a closest point of the plane $v'_i\langle b, c \rangle$ to $*$. As the line $v'_i\langle b \rangle$ separates $v'_i\langle b, c \rangle$ from $*$, x is a point of $v'_i\langle b \rangle$. Since $v_i\langle c \rangle$ separates $v'_i\langle b, c \rangle$ from $*$, a geodesic α , from x to $*$ intersects the line $v_i\langle c \rangle$, at say t . But the geodesic from v'_i to t (along the edge a^{-1} at v'_i followed by a path in $v_i\langle c \rangle$) is of shorter length than the subgeodesic of α connecting t and x , unless $x = v'_i$ (the Euclidean geodesic triangle (x, t, v'_i) has a right angle at v'_i). Similarly for w'_i . \square

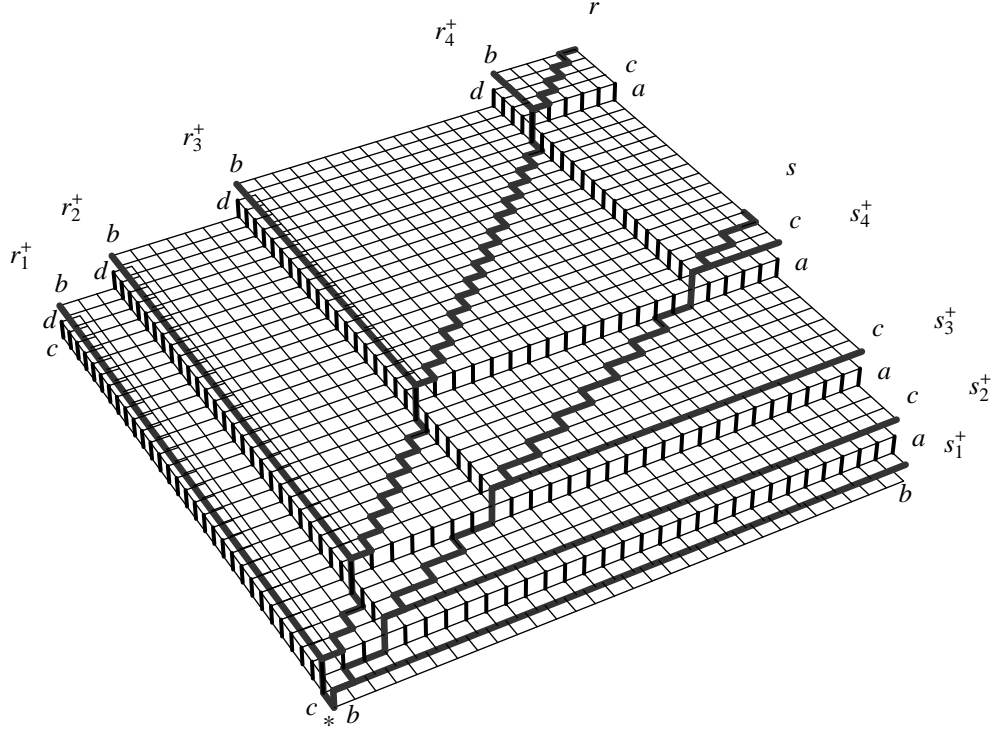


Figure 2: Detail of rays showing $\langle b, c \rangle$ cosets as parallel planes.

The geodesic lines $v'_n \langle b \rangle$ and $w'_n \langle c \rangle$ determine the points $b_n^{\pm\infty}$ and $c_n^{\pm\infty}$ respectively. Unfortunately, the geodesic from $*$ to v'_n followed by a non-trivial geodesic on the line $v'_n \langle b \rangle$ is not geodesic. Instead, for $n \geq 1$, define r_n^\pm (respectively s_n^\pm) to be the geodesic from $*$ to v'_n (respectively to w'_n) followed by the infinite geodesic in the positive/negative direction of the line $v'_n \langle c \rangle$ (respectively $w'_n \langle b \rangle$). In figure 3, r_i and r meet at v'_i , and s_i overlaps s on a b -edge at w'_i .

Claim 8 *The rays r_n^\pm and s_n^\pm are geodesic, r_n^+ (respectively r_n^-) have limit point $c_{n+1}^{+\infty}$ (respectively $c_{n+1}^{-\infty}$) and s_n^+ (respectively s_n^-) has limit point $b_n^{+\infty}$ (respectively $b_n^{-\infty}$).*

Proof: If x is a point of $v'_n \langle c \rangle$ in the image of r_n^+ , then any path from x to $*$ must cross the line $v'_n \langle b \rangle$. The shortest path from x to $v'_n \langle b \rangle$ is along the

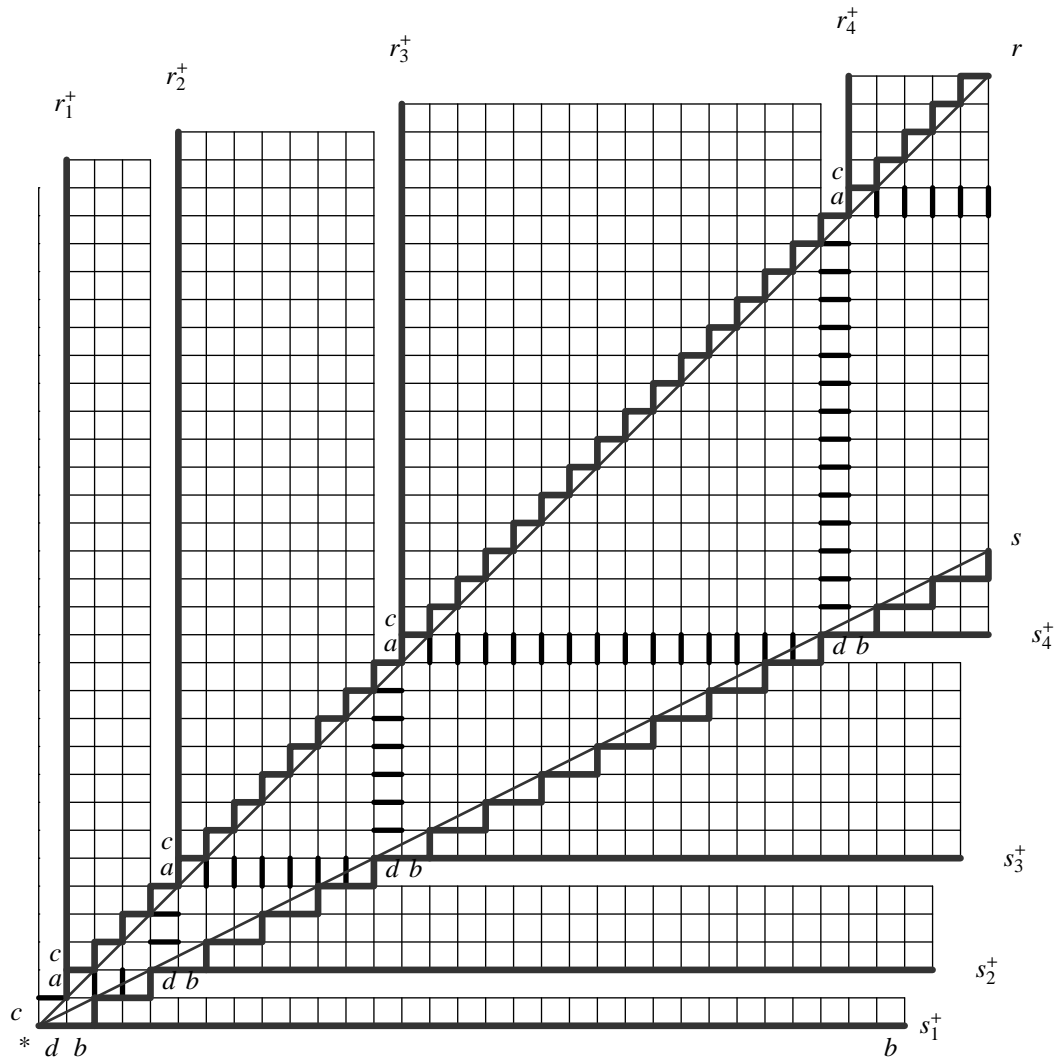


Figure 3: Detail of rays flattened into plane showing geodesics.

c -line through x (perpendicular to $v'_n\langle b \rangle$), so v'_n is the closest point of $v'_n\langle b \rangle$ to x . As v'_n is the closest point of $v'_n\langle b, c \rangle$ to $*$, r_n^+ is geodesic. Similarly for r_n^- and s_n^\pm .

By part 1) (respectively part 2)) of claim 4, r_n^\pm (respectively s_n^\pm) has $c_{n+1}^{\pm\infty}$ (respectively $b_n^{\pm\infty}$) as limit in ∂K . \square

Cayley graph geodesics need not stay a bounded distance from a CAT(0) space geodesic. The Cayley graph geodesics r and s are tracked by CAT(0) geodesics and have been introduced in the interest of defining rays in exact detail. To see the correct CAT(0) geometry we can conveniently flatten out into a plane the pertinent parts of the \mathcal{P}_n , and $\hat{\mathcal{P}}_n$ planes and B_n and C_n strips. This is illustrated in figure 3. A true CAT(0) geodesic with slope 1 passes through every other vertex in the r path. A CAT(0) geodesic with slope 1/2 passes through every third vertex in the s path. That is, r and s track CAT(0) space geodesics determining distinct points in the boundary, since these geodesics obviously diverge. Recall, points in ∂K are close if geodesic rays from $*$ converging to these points stay close for a long time. As the geodesic rays r_i^\pm pass through v'_n , and a geodesic for r passes within 1-unit of v'_n , $\{c_n^{\pm\infty}\}$ converges in ∂K to the limit point of r . Similarly, $\{b_n^{\pm\infty}\}$ converges to the limit point of s . By claim 5 the $c_i^{\pm\infty}$ and $b_i^{\pm\infty}$ are all distinct. In summary,

Claim 9 *Each of the Cayley graph paths r , s , are each within 2-units of some CAT(0) space geodesic. The points of the boundary determined by r and s as well as the points $b_n^{\pm\infty}, c_n^{\pm\infty}$ are all distinct. As ∂K is a metric space, there are disjoint closed neighborhoods of the limit points of r and s . For any neighborhood of the limit point of r , there is an n such that $c_i^{\pm\infty}$, the limit points of each r_{i-1}^\pm , for $i \geq n$, all lie in the neighborhood. Likewise, for any neighborhood of the limit point of s , there is an n such that $b_i^{\pm\infty}$, the limit points of each s_i^\pm , for $i \geq n$, all lie in the neighborhood. \square*

By claim 6, a path from c_1^+ to the limit point of r passes through one of the limit points of $b_i^{\pm\infty}$ and one of the limit points of $c_{i+1}^{\pm\infty}$, alternately for $i = 1, 2, 3, \dots$. But then this path alternately passes progressively closer to the limit points of r and s infinitely many times. The least upper bound of the inverse images of $c_i^{\pm\infty}$ is the same as the least upper bound of the inverse images of $b_i^{\pm\infty}$, and the image of this least upper bound would have to be both r and s . Thus there can be no path from c_1^+ to the limit point of r and ∂K is not path connected.

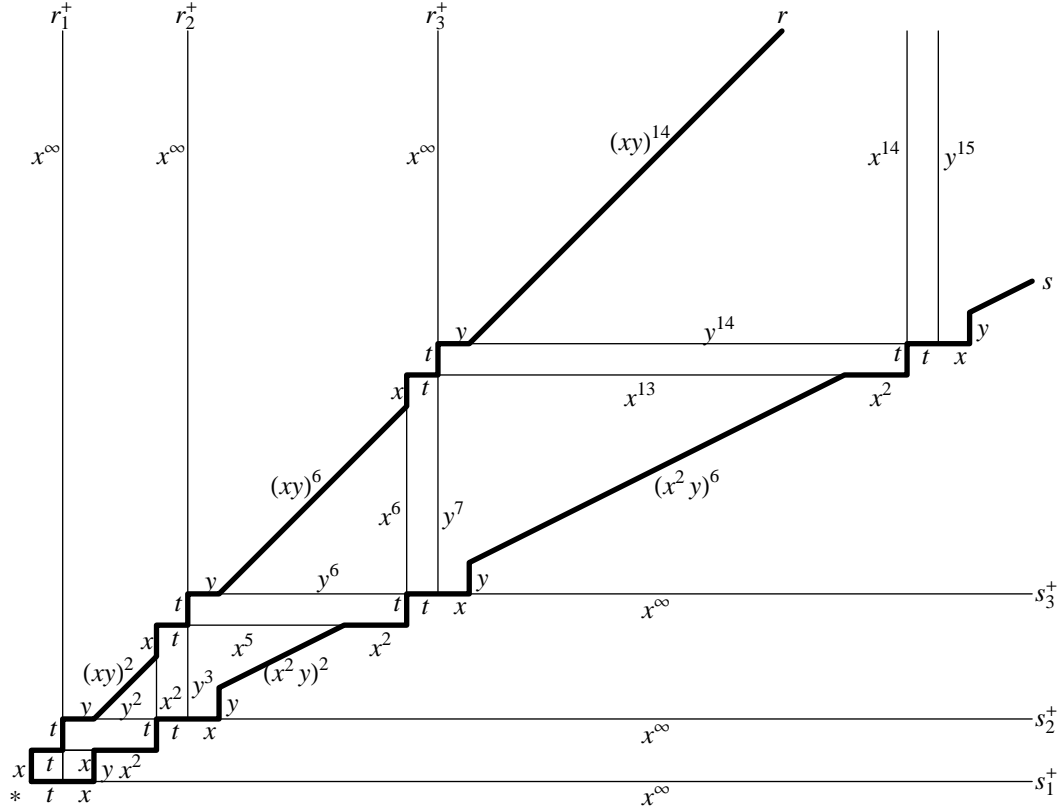


Figure 4: Schematic diagram of rays in $\langle t, x, y : [x, y] = 1, t^{-1}xt = y \rangle$.

Essentially the same proof applies to the group

$$P \equiv \langle t, x, y : [x, y] = 1, t^{-1}xt = y \rangle$$

with rays

$$r = xt^2y(xy)^2xt^2y(xy)^6xt^2y \dots = \prod_{i=1}^{\infty} (xy)^{k_i} xt^2y$$

and

$$s = txyx^2ttxy(x^2y)^2x^2ttxy(x^2y)^6x^2t \dots = \prod_{i=1}^{\infty} txy(x^2y)^{k_i} x^2t$$

where the exponents k_i are recursively defined by $k_1 = 0$ and, for all $i \geq 1$, $k_{i+1} = 2k_i + 2$ as before (see figure 4) to show this boundary for the group

is not path connected. This group is an HNN extension with base group $\langle x, y \rangle \cong \mathbb{Z} \times \mathbb{Z}$ associated subgroups $\langle x \rangle$ and $\langle y \rangle$ and stable letter t .

In [14], P is shown to be a “parabolic” semi-direct product of F_2 , the free group on two generators, and \mathbb{Z} . Another presentation for the group P is $\langle a, b, s : sas^{-1} = ab, sbs^{-1} = b \rangle$ (let $a = t$, $b = yx^{-1}$ and $s = x$ (so $y = bx = bs$) to derive the original presentation of P). For $n > 0$, the parabolic semi-direct products of F_2 and \mathbb{Z} are denoted P_n and have presentation $\langle a, b, s : sas^{-1} = ab^n, sbs^{-1} = b \rangle$. Note that the automorphism for P_n is the n^{th} iterate of that for $P = P_1$. This implies every parabolic semi-direct product of F_2 and \mathbb{Z} is isomorphic to a subgroup of finite index of P . Hence every CAT(0) boundary for P is also a CAT(0) boundary for every parabolic semi-direct product of F_2 and \mathbb{Z} . If S is a semi-direct product of F_2 and \mathbb{Z} and some iterate of the corresponding automorphism of F_2 is inner, then S has a subgroup of finite index isomorphic to $F_2 \times \mathbb{Z}$. If S acts on the CAT(0) space Y , then the “min set” of \mathbb{Z} in Y splits as a metric product $F \times \mathbb{R}$ and is quasi-dense in Y . Hence ∂Y is homeomorphic to the (path connected) suspension of a Cantor set (see [3]). If an automorphism of F_2 is not parabolic and no iterate of the automorphism is inner, then the automorphism is called hyperbolic. Every semi-direct product of F_2 and Z contains a $\mathbb{Z} \times \mathbb{Z}$ subgroup, and so hyperbolic automorphisms do not give word hyperbolic semi-direct products (the automorphism must take the commutator for F_2 to a conjugate of itself or its inverse). In [18], K. Ruane shows the only CAT(0) boundary possible for a hyperbolic semi-direct product of F_2 and \mathbb{Z} is the Sierpinski carpet, a compact, connected and locally path connected metric space. In [1], T. Brady shows that every semi-direct product of F_2 and \mathbb{Z} acts geometrically on a CAT(0) space, but in [10], Gersten constructs semi-direct products of F_3 , the free group of rank 3, and \mathbb{Z} which do not act geometrically on any CAT(0) space. Sumarizing,

Theorem *There is a CAT(0) space X with connected, but non-path connected boundary such that every parabolic semi-direct product of the free group of rank 2 and \mathbb{Z} acts geometrically on X .*

If an iterate of an automorphism of the free group of rank 2 is inner, the only possible CAT(0) boundary for the resulting semi-direct product is the (path connected and non-locally connected) suspension of a Cantor set.

(Ruane [18]) If an automorphism of the free group of rank 2 is hyperbolic, the only possible boundary for the resulting semi-direct product is the (connected and locally path connected) Sierpinski carpet.

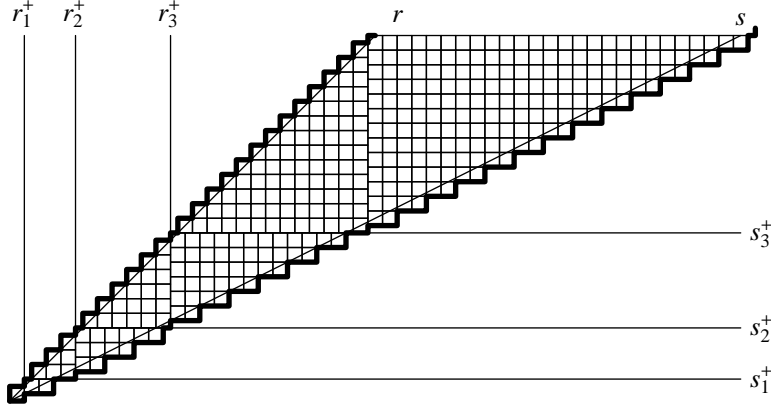


Figure 5: Schematic of rays and flattenings for $\langle x, y : [x^2, y^2] = 1 \rangle$.

Our final example of this type is the group $Q \equiv \langle x, y : [x^2, y^2] = 1 \rangle$. This group decomposes as $\langle x \rangle *_{\langle x^2 \rangle} \langle x^2, y^2 \rangle *_{\langle y^2 \rangle} \langle y \rangle$. Define $m_0 = 1$, $m_{i+1} = 4m_i + 2$ for $i \geq 0$, or in closed form

$$m_i = \frac{5 \cdot 4^i - 2}{3}$$

so $m_0 = 1$, $m_1 = 6$, $m_2 = 26$, $m_3 = 106$, $m_4 = 426 \dots$

Take rays

$$\begin{aligned} r &= \prod_{i=0}^{\infty} (b^2 a^2)^{m_i} b a (b^2 a^2)^{2m_i+1} b a \\ &= b^2 a^2 b a (b^2 a^2)^3 b a (b^2 a^2)^6 b a (b^2 a^2)^{13} b a (b^2 a^2)^{26} b a (b^2 a^2)^{53} b a \dots \end{aligned}$$

and

$$\begin{aligned} s &= a^2 b \prod_{i=0}^{\infty} (a^4 b^2)^{m_i} a^3 b (a^4 b^2)^{2m_i+1} a b \\ &= a^2 b a^4 b^2 a^3 b (a^4 b^2)^3 a b (a^4 b^2)^6 a^3 b (a^4 b^2)^{13} a b (a^4 b^2)^{26} a^3 b (a^4 b^2)^{53} a b \dots \end{aligned}$$

As in the previous examples Q has connected, but non-path connected boundary (see figure 5).

We end this section with some observations about the (two dimensional) 1-relator group $Q \equiv \langle x, y : [x^2, y^2] = 1 \rangle$. The group Q is a semi-direct

product of F_3 and \mathbb{Z} where the subgroup $Q_1 = \langle xy^{-1}, x^{-1}y, x^2y^{-2} \rangle$ is free of rank 3, and normal in Q with quotient \mathbb{Z} . If A is an arbitrary semi-direct product of a finitely generated non-abelian free group with \mathbb{Z} , then as in the case with Q and all parabolic semi-direct products of F_2 and \mathbb{Z} , A may have boundary which is not path connected. It may be that A is word hyperbolic (see for example [2]) or A is a hyperbolic semi-direct product of F_2 and \mathbb{Z} , in either case ∂A is connected and locally path connected. If $A = F_n \times \mathbb{Z}$ for some $n > 1$, then the suspension of a Cantor set (a path connected, but non-locally connected space) is the only CAT(0) boundary for A . Let α be the automorphism of Q_1 induced by conjugation by x . No iterate of α , is inner as can be easily checked, but the fact that a boundary for Q is not path connected also implies no iterate of α is inner (if an iterate of α were inner, then the only possible CAT(0) boundary would be the suspension of a Cantor set).

Recall that Gersten [10] has constructed semi-direct products of F_3 and \mathbb{Z} which do not act geometrically on any CAT(0) space. Hence, when they exist, CAT(0) boundaries of semi-direct products of finitely generated free groups and \mathbb{Z} can realize a wide range of behavior with respect to path connectedness and local connectedness.

3 Principal components of the constructions

We analyze the key components of the proof that H , Q and P have a boundary that is not path connected. When H is written as $\langle A \rangle *_D \langle B \rangle *_E \langle C \rangle$ where $A = \langle a, b \rangle$, $B = \langle b, c \rangle$, $C = \langle c, d \rangle$, $D = \langle b \rangle$ and $E = \langle c \rangle$ we have used the fact the limit sets determined by D and E are disjoint. Correspondingly, for $Q = \langle x, y : [x^2, y^2] = 1 \rangle = \langle x \rangle *_{\langle x^2 \rangle} \langle x^2, y^2 \rangle *_{\langle y^2 \rangle} \langle y \rangle$, the limit sets of the amalgams $\langle x^2 \rangle$ and $\langle y^2 \rangle$ are disjoint and for $P = \langle t, x, y : [x, y] = 1, t^{-1}xt = y \rangle$ the limit sets of the associated subgroups $\langle x \rangle$ and $\langle y \rangle$ are disjoint. If one considers $H_1 \equiv H \times \mathbb{Z}_t$ where \mathbb{Z}_t is infinite cyclic with generator t , then an analogous attempt at our constructions would lead to an H_1 -action on a CAT(0) space with boundary the suspension of our boundary for H . In particular, we would obtain a path connected boundary for H_1 . Where an argument corresponding to the one showing H has a non-path connected boundary would fail is in the fact that the corresponding amalgamated subgroups for H_1 would be $\langle b, t \rangle$ and $\langle c, t \rangle$. The limit sets determined by these groups overlap at the two limit points determined by $\langle t \rangle$.

It was necessary to construct geodesic rays r and s that converge to different points of the boundary and pass through the same infinite collection of $\langle B \rangle$ cosets. This means the corresponding rays in the Bass-Serre tree for our decomposition traverse the same infinite collection of edges, which alternate between D and E cosets. We used the fact that the set $A - D$ (respectively $C - E$) contained an element a (respectively d) that commuted with D (respectively E), in order to build strips that together with portions of the B cosets traversed by r and s formed “flat” subsets of the CAT(0) space Y . In particular, if say A , B and C are replaced by hyperbolic closed surface groups, and the resulting group is word hyperbolic, then its boundary is in fact locally connected and path connected by [21]. A final important aspect of the geometry of our examples was the fact that r and s alternately entered B cosets at points which are the closest points of the coset to the identity vertex. This fact ensured that our $r_n^{\pm\infty}$ and $s_n^{\pm\infty}$ constructions were geodesic and that the limit sets of $r_n^{\pm\infty}$ (respectively $s_n^{\pm\infty}$) converge to the limit point of r (respectively s).

Our examples are in some sense the smallest we can find for which our techniques succeed. At this time, the only general results we are able to obtain are simply results that faithfully embed our examples with the prescribed geometry (or simple variants) into groups. For this reason we forgo such statements. General questions are postponed until the last section.

4 Higher dimensional examples

We use [19] as a basic reference for this section. Let $S(\partial X)$ be the suspension of ∂X . As noted earlier $H \times \mathbb{Z}$ is simply connected at infinity and has boundary $S(\partial X)$. Let U_1 be the upper cone in S on ∂X , U_2 be the lower cone on ∂X and U_0 the base copy of ∂X in S (so that $U_1 \cap U_2 = U_0$). The reduced Mayer-Vietoris theorem implies that the following sequence is exact:

$$H_1(U_1) \oplus H_1(U_2) \rightarrow H_1(S) \rightarrow H_0^\#(U_0) \rightarrow H_0^\#(U_1) \oplus H_0^\#(U_2)$$

As U_1 and U_2 are contractible, the rank of $H_1(S)$ is one less than the number of path components of U_0 . In particular $\pi_1(S)$ is non-trivial.

The group $H \times \mathbb{Z}^2$ acts geometrically on $X \times \mathbb{R}^2$, with boundary S_2 , the suspension of S . Note that S_2 is 1-connected (Van Kampen’s theorem). By the Hurewicz theorem, $\pi_2(S_2)$ is isomorphic to $H_2(S_2)$. As above, we let V_1

and V_2 be the upper and lower cone on V_0 , the base copy of S in S_2 . The Mayer-Vietoris theorem implies that the following sequence is exact:

$$H_2(V_1) \oplus H_2(V_2) \rightarrow H_2(S_2) \rightarrow H_1(V_0) \rightarrow H_1(V_1) \oplus H_1(V_2)$$

As V_1 and V_2 are contractible, $H_2(S_2)$ is isomorphic to $H_1(V_0) = H_1(S)$. Since the rank of $H_1(S)$ is one less than the cardinality of the number of path components of ∂X , we see that while $H \times \mathbb{Z}^2$ is 2-connected at infinity and the boundary, S_2 , of $H \times \mathbb{Z}^2$ is 1-connected, $\pi_2(S_2)$ is non-trivial.

Inductively, we see $H \times \mathbb{Z}^n$ is n -connected at infinity, with boundary S_n , the n -fold suspension of ∂H (with right angled geometry) and that $\pi_i(S_n) = 0$ for $i \in \{0, \dots, n-1\}$, but $\pi_n(S_n) \neq 0$.

With a slight modification of our presentation of H a right angled Coxeter group can be constructed with the same pathology at infinity as H . We simply replace a and d by order 2 elements and exchange the cyclic groups $\langle c \rangle$ and $\langle d \rangle$ with copies of $\mathbb{Z}_2 * \mathbb{Z}_2$. A presentation for this group is

$$\langle a, b_1, b_2, c_1, c_2, d : a^2 = b_i^2 = c_i^2 = d^2 = 1 = [a, b_i] = [b_i, c_j] = [c_k, d] \rangle$$

5 A second construction

The point of this section is to produce a second construction for simply connected at infinity, 1-ended CAT(0) groups with non-simply connected boundary. Our next example, G , has the following presentation:

$$\langle x_1, y_1, t, s, x_2, y_2, z_1, z_2 : \\ [x_i, y_i] = [t, s] = [z_i, s] = [z_i, x_i] = [z_i, y_i] = [z_i, t] = 1, t^{-1}x_it = y_i \rangle$$

This group, should be thought of as the amalgamated product:

$$A_1 *_{\mathbb{Z}_s \oplus \mathbb{Z}_t} A_2$$

where A_i is presented as the direct product $B_i \times \mathbb{Z}_i$ by

$$\langle x_i, y_i, t, s : t^{-1}x_it = y_i, [x_i, y_i] = [t, s] = 1 \rangle \times \langle z_i \rangle$$

and the amalgamated subgroup, $\mathbb{Z}_s \oplus \mathbb{Z}_t$, is $\langle s, t : [s, t] = 1 \rangle$. The group B_i is easily seen to be 1-ended so A_i is simply connected at infinity. The product of two simply connected at infinity groups amalgamated along a

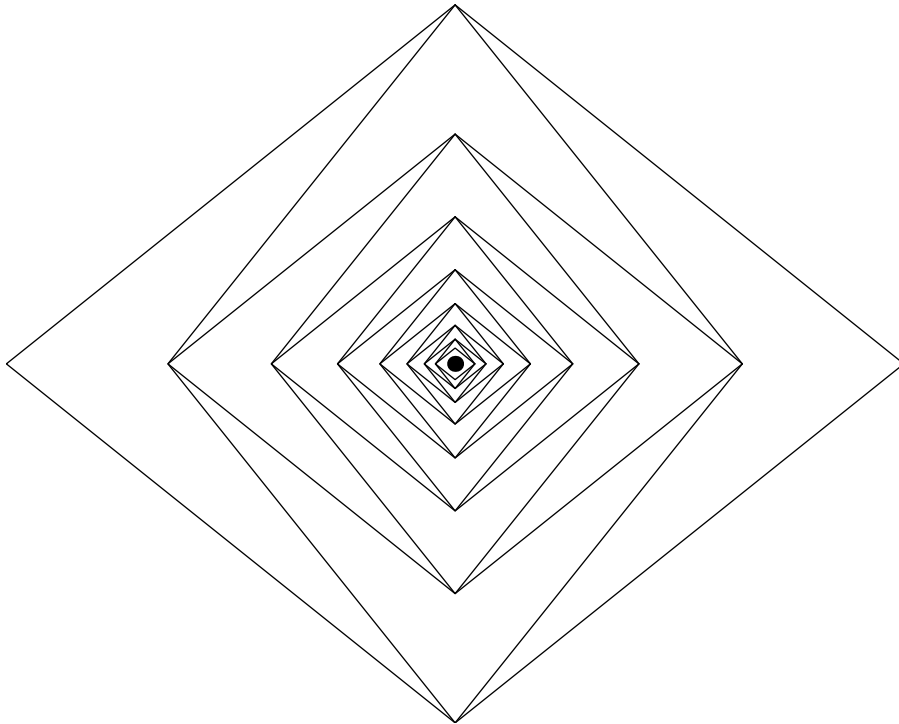


Figure 6: Schematic diagram of Hawaiian earrings in $\langle t, x, y : [x, y] = 1, t^{-1}xt = y \rangle$.

finitely presented 1-ended group is simply connected at infinity [11] and so G is simply connected at infinity.

This brings us to the geometry of G . First of all, B_i is $C_i *_{\mathbb{Z}_t} (\mathbb{Z}_t \oplus \mathbb{Z}_s)$ where C_i has the presentation $\langle x_i, y_i, t : t^{-1}x_it = y_i, [x_i, y_i] = 1 \rangle$. The amalgamated subgroup \mathbb{Z}_t is $\langle t \rangle$ and the factor $\mathbb{Z}_t \oplus \mathbb{Z}_s$ is $\langle t, s : [t, s] = 1 \rangle$. In boundaries of the CAT(0) group C_i , one finds “Hawaiian earrings” arising from the t -translates of the circles generated by the $\mathbb{Z} \oplus \mathbb{Z}$ subgroup $\langle x_i, y_i \rangle$. Two consecutive circles (translates by t^j and t^{j+1}) share two points. The circles corresponding to translates by t^n for $n > 0$, limit to t^∞ in the boundary. The point t^∞ should be thought of as the vertex of a Hawaiian earring (see figure 6).

If we consider C_1 and C_2 we see two Hawaiian earrings joined at their common vertex t^∞ . The effect of considering $B_i \times \mathbb{Z}_i$ is to suspend the boundary of B_i . This produces two suspended Hawaiian earrings joined at

the point t^∞ , faithfully in ∂G . Even though the suspended Hawaiian earrings are separately simply connected, together they are not. (There is no Van Kampen Theorem here!)

Hence in exactly the same way as with the first example, for $n \geq 1$ and $i \in \{0, \dots, n\}$, $\pi_i^\infty(G \times \mathbb{Z}^n) = 0$. For $i \in \{1, \dots, n-1\}$, $\pi_i(\partial G \times \mathbb{Z}^n) = 0$, but $\pi_n(\partial G \times \mathbb{Z}^n) \neq 0$.

As noted earlier, the base group of our construction here $C = \langle x, y, t : t^{-1}xt = y, [x, y] = 1 \rangle$ has non-path connected boundary, but our construction of a simply connected at infinity group with non-simply connected boundary is not the same as before. It is this second construction that is of interest here. This is discussed again after question 1 of the next section.

6 Questions and more observations

Word hyperbolic groups have unique boundary. By [21], 1-ended word hyperbolic groups have locally connected boundary and hence path connected and locally path connected boundary. (By classical topology the boundary is a Peano space.)

We give a discussion of n -connectedness at infinity for CAT(0) spaces. Analogous statements can be made about word hyperbolic groups. Let X be a non-compact proper CAT(0) space with cone-topology boundary ∂X . Choose base points $x_0 \in X$ and $e_0 \in \partial X$. The geodesic ray starting at x_0 and ending at e_0 is the *base ray*, denoted $\omega : [0, \infty) \rightarrow X$. A convenient basic system of neighborhoods of ∂X in X is $\{U_k\}$, where $U_k \equiv X - B_k(x_0)$, the complement of the ball of radius k . As explained in Ross Geoghegan's book [7], there are three ways of making precise the vague notion of " n -connectedness of ∂X " or " n -connectedness of X at ∞ ".

(1) ∂X is an n -connected space.

(2) For all $p \leq n$ every proper map $f : S^p \times [0, \infty) \rightarrow X$ extends to a proper map $F : B^{p+1} \times [0, \infty) \rightarrow X$. One says " X is strongly n -connected at ∞ ".

(3) For all $p \leq n$ and all k there exists $l \geq k$ such that singular spheres in $X - B_l(x_0)$ bound singular balls in $X - B_k(x_0)$. One says " X is n -connected at ∞ ".

These are in decreasing order of strength: (1) implies (2) implies (3).

For $p \leq n$, we have an inverse sequence of homotopy groups (sets in dimension 0).

$$(*)\{\pi_p(U_1, w_1) \leftarrow \pi_p(U_2, w_2) \leftarrow \cdots\}$$

where $w_n = \omega(n+1)$ and the bonding morphisms are defined using the usual change of base points along ω . We have:

(3) holds iff the sequence $(*)$ is pro-trivial for all $p \leq n$; i.e. given i there exists $j \geq i$ such that the image of $\pi_p(U_j, w_j) \rightarrow \pi_p(U_i, w_i)$ is trivial.

(2) holds iff (3) holds and $\lim_{\leftarrow}^1(*)$ is trivial in dimension $p = n + 1$.

The following statement is well-known and an easy exercise in geometric point-set topology. If ∂X is locally n -connected (LC^n), and X is n -connected at ∞ then ∂X is n -connected.

For $n = 0$, (1) says that ∂X is path connected, (2) says that any two proper rays in X are properly homotopic, and (3) says that X has one end, or, equivalently, that ∂X is connected. For a further discussion of these issues, see the introduction to [8] and the Appendix to [9].

The following is a restatement of the main result of [21]: If the word hyperbolic group G is 0-connected at infinity (i.e. 1-ended), then $\partial(G)$ is locally 0-connected (i.e. locally path connected).

Question 1 If a word hyperbolic group G is n -connected at infinity

1. is $\partial(G)$ locally n -connected?
2. is $\partial(G)$ n -connected?

As noted, an affirmative answer to part 1 of Question 1 implies part 2. Alternatively, our construction of G in §5 seems to give some promise for constructing a word hyperbolic group that is simply connected at infinity, but with non-simply connected boundary. See [13], [14] and [15] for a discussion of Coxeter groups and general CAT(0) groups with non-locally connected boundary.

A more general type of question should be considered. Given a group G acting geometrically on a CAT(0) space X , there is a natural map $f_{(X,\omega)} : \pi_1(\partial X, \omega) \rightarrow \pi_1^\infty(X, \omega) \equiv \pi_1^\infty(G, \omega)$ where $\pi_1^\infty(X, \omega)$ is the fundamental group at infinity of X with base ray ω , a geodesic in X . Similarly for any word hyperbolic group G there is a natural map $f_{(G,\omega)} : \pi_1(\partial G, \omega) \rightarrow \pi_1^\infty(G, \omega)$. (See [7] and [9] for a complete discussion of $\pi_1^\infty(G)$ for any finitely presented group G .)

Question 2 When is $f_{(X,\omega)}$ ($f_{(G,\omega)}$) injective?

If G is a word hyperbolic group it makes sense to ask if $f_{(G,\omega)}$ is always injective. Observe that, if G is word hyperbolic and ∂G is locally 1-connected

then $f_{(G,\omega)}$ is injective. If G acts geometrically on the CAT(0) space X and ∂X is locally 1-connected then $f_{(X,\omega)}$ is injective. In [4], G.R. Conner and H. Fischer consider question 2 in a more general setting. In particular, they show the map is π_1 -injective when the boundary is 1-dimensional.

Question 3 Given two boundaries for a 1-ended CAT(0) group, must the homotopy (homology) groups of these spaces agree? In particular, must they agree at the π_0 -level.

References

- [1] T. Brady, *Complexes of non-positive curvature for extensions of F_2 by \mathbb{Z}* , Topology Appl. **63** (1995), 267-275.
- [2] M. Bestvina and M. Feighn, *A combination theorem for negatively curved groups*, J. Differential Geom. **35** (1992), 85-101.
- [3] M.R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundle Math. Wiss. **319**, Springer, Berlin, 1999.
- [4] G.R. Conner and H. Fischer, *The fundamental group of a visual boundary versus the fundamental group at infinity*, Topology Appl. **129** (2003), no. 1, 73-78.
- [5] C.B. Croke and B. Kleiner, *Spaces with nonpositive curvature and their ideal boundaries*, Topology **39** (2000) no. 3, 549-556.
- [6] C.B. Croke and B. Kleiner, Personal communication.
- [7] R. Geoghegan, *Topological methods in group theory*, Preprint.
- [8] R. Geoghegan and J. Krasinkiewicz, *Empty components in strong shape theory*, Topology Appl. **41** (1991), 213-233.
- [9] R. Geoghegan and M.L. Mihalik, *The fundamental group at infinity*, Topology **35** (1996) no. 3, 655-669.
- [10] S.M. Gersten, *The automorphism group of a free group is not a CAT(0) group*, Proc. Amer. Math. Soc. **121** (1994), 999-1002.

- [11] B. Jackson, *End invariants of amalgamated free products*, J. of Pure and Appl. Alg. **23** (1982), 233-242.
- [12] F.E.A. Johnson, *Manifolds of homotopy type $K(\pi, 1)$, II*, Proc. Cambridge Philos. Soc. **75** (1974), 165-173.
- [13] M.L. Mihalik and K. Ruane, *Non-locally connected boundaries of $CAT(0)$ groups*, J. London Math. Soc. (2) **60** (1999), 757-770.
- [14] M.L. Mihalik and K. Ruane, *$CAT(0)$ HNN-extensions with non-locally connected boundary*, Topology Appl. **100** (2001), 83-98.
- [15] M.L. Mihalik, K. Ruane and S. Tschantz, *Local connectivity of right-angled Coxeter group boundaries*, To appear in J. Group Theory.
- [16] M.L. Mihalik and S. Tschantz, *Semistability of Amalgamated Products and HNN-Extensions*, Memoirs of the AMS, **98**, no. 471, 1992.
- [17] M.L. Mihalik and S. Tschantz, *One relator groups are semistable at infinity*, Topology **31** (1992), 801-804.
- [18] K. Ruane, *Boundaries of truncated hyperbolic space*, Topology Proc. **29** (2005), no. 1, 317-331
- [19] E. Spanier, *Algebraic Topology*, McGraw-Hill Book Co., New York-Toronto, Ont.-London (1966).
- [20] J.R. Stallings, *The piecewise-linear structure of Eculidean space*, Proc. Cambridge Philos. Soc. **58** (1962), 481-487.
- [21] G. Swarup, *On the cut point conjecture*, Electron. Res. Annouc. Amer. Math. Soc. **2** (1996) no. 2, 98-100.

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