§13.4 Cross Product

(In this section, we will consider vectors only in 3-dimensional space)

<u>Definition</u>: If $\underline{a} = \langle a_1, a_2, a_3 \rangle$ & $\underline{b} = \langle b_1, b_2, b_3 \rangle$, then the <u>cross product of $\underline{a}$ & $\underline{b}$</u> denoted by $\underline{a} \times \underline{b}$, is a <u>vector</u> given by

$$\underline{a} \times \underline{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

<u>Notation</u>: (i) Determinant of order 2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

(i) Determinant of order 3

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

<u>Formula</u>: If $\underline{a} = \langle a_1, a_2, a_3 \rangle$ & $\underline{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\underline{a} \times \underline{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \underline{i} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} \underline{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \underline{k}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

<u>Properties</u>: If $\underline{a}$ & $\underline{b}$ are vectors & c is a scalar, then

1) $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$

2) $(c\underline{a}) \times \underline{b} = c(\underline{a} \times \underline{b}) = \underline{a} \times (c\underline{b})$

3) $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$

4) $(\underline{a} + \underline{b}) \times \underline{c} = \underline{a} \times \underline{c} + \underline{b} \times \underline{c}$

5) $\underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c}$

6) $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$

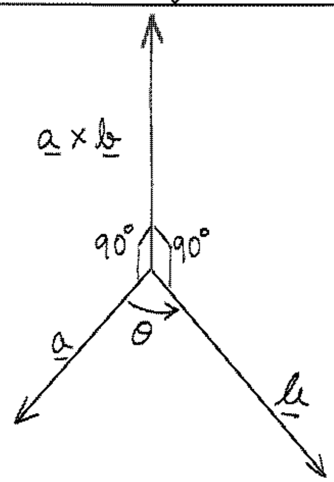
Theorem: $\underline{a} \times \underline{b}$ is orthogonal to both $\underline{a}$ & $\underline{b}$

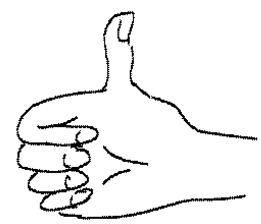
Theorem: If θ is the angle between $\underline{a}$ & $\underline{b}$ such that $0 \leq \theta \leq \pi$, then

$$|\underline{a} \times \underline{b}| = |\underline{a}|\,|\underline{b}| \sin(\theta)$$

Thus, if $\underline{a}$ & $\underline{b}$ are parallel, then $\underline{a} \times \underline{b} = \underline{0}$

Direction of $\underline{a} \times \underline{b}$ with respect to $\underline{a}$ & $\underline{b}$



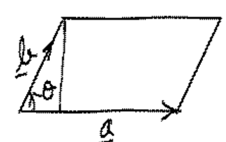


Facts:-

(i) Three vector $\underline{a}, \underline{b}$ & $\underline{c}$ (with the same initial point) are **coplanar** if and only if $\boxed{\underline{a} \cdot (\underline{b} \times \underline{c}) = 0}$

(ii) Four points P, Q, R & S are **coplanar** if and only if $\boxed{\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) = 0}$

Formula:- (i) Area of the parallelogram formed by $\underline{a}$ & $\underline{b}$ $= |\underline{a} \times \underline{b}|$



(ii) Volume of the parallelopiped formed by $\underline{a}, \underline{b}$ & $\underline{c}$ $= |\underline{a} \cdot (\underline{b} \times \underline{c})|$