Math 175 Exam 1

Name(print): Key.

Problem 1:	/15
Problem 2:	/20
Problem 3:	/15
Problem 4:	/15
Problem 5:	/10
Problem 6:	/25
Total:	/100

average 76

This is a paper-based exam. No calculator is allowed for this exam. To get full points, you must justify your answer(s).

1. (15 points) Given points $P(\sqrt{3}, \pi/2, \pi/6)$ (in spherical) and $Q(1, \pi/6, \sqrt{5})$ (in cylindrical coordinates). Find the area of the triangle formed by O, P and Q.

$$P\left(\underset{\rho}{\sqrt{3}}, \underset{\theta}{\tfrac{\pi}{2}}, \underset{\varphi}{\tfrac{\pi}{6}}\right) \to \begin{cases} x = \rho\sin\varphi\cos\theta = 0 \\ y = \rho\sin\varphi\sin\theta = \frac{\sqrt{3}}{2} \\ z = \rho\cos\varphi = 3/2 \end{cases} \to P\left(0, \frac{\sqrt{3}}{2}, \frac{3}{2}\right) \text{ — 4 pt}$$

$$Q\left(\underset{\rho}{1}, \underset{\theta}{\tfrac{\pi}{6}}, \underset{z}{\sqrt{5}}\right) \to \begin{cases} x = \rho\cos\theta = \sqrt{3}/2 \\ y = \rho\sin\theta = 1/2 \\ z = z = \sqrt{5} \end{cases} \to Q\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{5}\right) \text{ — 4 pt}$$

$$\text{area} \underset{\text{2pt}}{=} \frac{1}{2}\left|\vec{OP} \times \vec{OQ}\right| \underset{\text{4pt}}{=} \frac{1}{2}\left\langle \frac{5\sqrt{3}}{2} - \frac{1}{4}, \frac{3\sqrt{3}}{4}, -\frac{3}{4}\right\rangle \underset{\text{2pt}}{=} \frac{1}{2}\sqrt{\left(\frac{5\sqrt{3}}{2} - \frac{3}{4}\right)^2 + \left(\frac{3\sqrt{5}}{4}\right)^2 + \left(-\frac{3}{4}\right)^2}$$

2. (20 points) Given the planes $2x + y - z = 0$ and $x + 2y + 3z = 3$.

(a) find the cosine of the angle between the two plane.

$$\vec{n}_1 = \langle 2, 1, -1\rangle \qquad \vec{n}_2 = \langle 1, 2, 3\rangle \text{ — 4 pt}$$

$$\cos\theta \underset{\text{4pt}}{=} \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} \underset{\text{2pt}}{=} \frac{2+2-3}{\sqrt{6} \cdot \sqrt{14}} = \frac{1}{2\sqrt{21}}.$$

(b) find the vector equation of the intersecting line of the two planes.

$$\vec{v}_0 = \vec{n}_1 \times \vec{n}_2 = \langle 5, -7, 3\rangle. \text{ — 5 pt}$$

$$\begin{cases} 2x + 2y - z = 0 \\ x + 2y + 3z = 3 \end{cases} \xrightarrow{x = 0} \left(0, \frac{3}{5}, \frac{3}{5}\right) \text{ — 3 pt}$$

$$\vec{v} = \left\langle 0, \frac{3}{5}, \frac{3}{5}\right\rangle + t\langle 5, -7, 3\rangle \text{ — 2 pt}$$

3. (15 points) Show that the distance between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

Assume $a \neq 0$, then $P\left(-\frac{d_1}{a}, 0, 0\right)$ is on the first plane. — 5pt

distance of the planes

2pt $\stackrel{\text{}}{=}$ distance from P to the second plane

$$\stackrel{7pt}{=} \frac{\left|a\cdot\left(-\frac{d_1}{a}\right) + b\cdot 0 + c\cdot 0 + d_2\right|}{\sqrt{a^2+b^2+c^2}} \stackrel{1pt}{=} \frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}.$$

4. (15 points) Using definition of derivatives of vector functions to prove the following formula:

$$\frac{d}{dt}[f(t)\vec{v}(t)] = f'(t)\vec{v}(t) + f(t)\vec{v}\,'(t).$$

Let $\vec{v}(t) = \langle x(t), y(t), z(t)\rangle$. — 3pt

$$\frac{d}{dt}\left(f(t)\,\vec{v}(t)\right) = \frac{d}{dt}\left(\langle f(t)x(t), f(t)y(t), f(t)z(t)\rangle\right)$$

$$\stackrel{3pt}{=} \langle (f(t)x(t))', (f(t)y(t))', (f(t)z(t))' \rangle$$

$$\stackrel{3pt}{=} \langle f'(t)x(t) + f(t)x'(t), f'(t)y(t) + f(t)y'(t), f'(t)z(t) + f(t)z'(t)\rangle$$

$$\stackrel{3pt}{=} \langle f'(t)x(t), f'(t)y(t), f'(t)z(t)\rangle + \langle f(t)x'(t), f(t)y'(t), f(t)z'(t)\rangle$$

$$\stackrel{3pt}{=} f'(t)\,\vec{v}(t) + f(t)\,\vec{v}\,'(t).$$

5. (10 points) The parabola $z = 4y^2$, $x = 0$ is rotated about the z-axis. Write an equation of the resulting surface.

Observe that ① when $y=0$, we have parabola $z=4x^2$. — 3pt

② the surface is an elliptic paraboloid, thus $z=ay^2+bx^2$.

Since $z=4x^2$ when $y=0$ & $z=4y^2$ when $x=0$, $a=b=4$. That is $z=4x^2+4y^2$. 3pt

2+2 pt

6. (25 points) Solve the following problems:

(a)

$$\vec{r}(t) = \cos t\ \vec{i} + 2t^2\ \vec{j} + 3\sin 2t\ \vec{k},$$

(i) find $\lim_{t\to 2}\vec{r}(t)$.

$\lim_{t\to 2}\vec{r}(t) = \cos 2\cdot\vec{i} + 8\vec{j} + 3\sin 4\,\vec{k}$ — 6pt

(ii) find $\vec{r}\,'(t)$.

$\vec{r}\,'(t) = -\sin t\cdot\vec{i} + 4t\cdot\vec{j} + 6\cos 2t\cdot\vec{k}$ — 6pt

(b)

$$\int (\frac{4}{t^2+2}\ \vec{j} + \frac{2t}{1+t^2}\ \vec{k})\ dt$$

2pt

$= \left(\int \frac{4}{t^2+2}\,dt\right)\vec{j} + \left(\int \frac{2t}{1+t^2}\,dt\right)\vec{k}$

$= 4\cdot\frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}\,\vec{j} + \ln|1+t^2|\,\vec{k} + \vec{C}$ — 3pt

where $\vec{C}$ is any constant vector.

4pt

4pt