Herbrand's Theorem, Skolemization, and Proof Systems for First-Order Łukasiewicz Logic[⋆]

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Abstract. An approximate Herbrand theorem is established for first-order infinite-valued Łukasiewicz Logic and used to obtain a proof-theoretic proof of Skolemization. These results are then used to define proof systems in the framework of hypersequents. In particular, a calculus lacking cut-elimination is defined for the first-order logic characterized by linearly ordered MV-algebras, a cut-free calculus with an infinitary rule for the full first-order Łukasiewicz Logic, and a cut-free calculus with finitary rules for its one-variable fragment.

1 Introduction

Infinite-valued Łukasiewicz Logic was introduced for philosophical reasons by Jan Łukasiewicz in the 1930s [17], and is among the most important and widely studied of all non-classical logics. In particular, it is considered, along with Gödel Logic and Product Logic, one of the fundamental "t-norm based" fuzzy logics most suitable for formalizing reasoning in the context of vagueness (see [13] for details). The algebraic semantics of this logic, MV-algebras, are also the subject of intensive research in Algebra, having deep connections with abelian ℓ-groups with strong unit and, via McNaughton's theorem [18], continuous piecewise linear functions on $[0, 1]$. The monograph [12] provides an in-depth treatment of these and many other topics.

The first-order counterpart $\forall$Ł of Łukasiewicz Logic is defined by generalizing the classical interpretations of the quantifiers $\forall$ and $\exists$ to infima and suprema in $[0, 1]$.[3] However, unlike the situation in Classical Logic, the valid formulas of $\forall$Ł are not recursively enumerable, a result obtained by Scarpellini [27] and later sharpened by Ragaz to Π_2-completeness [25]. Axiomatizations of $\forall$Ł with infinitary rules have been obtained nevertheless; by Hay [15], Belluce and Chang [8, 7] (see also Mostowski [21]),

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[3] Note that it is also common in the literature (see e.g. [13]) to use "first-order Łukasiewicz Logic" to refer to the extension of an axiomatization of the propositional logic with suitable axioms and rules for quantifiers (such as may be found e.g. in Figure 1). However, such a system is sound but cannot be complete with respect to $\forall$Ł. We therefore emphasize this key terminological difference to the reader now with the hope of avoiding confusion later on.

and Hájek [13]. Various fragments of $\forall$Ł have also been investigated. Validity and satisfiability in the one-variable fragment were proved to be decidable by Rutledge [26], while satisfiability for the monadic fragment was proved to be Π_1-complete by Ragaz [25] (the complexity of validity for this fragment remains an interesting open problem). Also of interest are (decidable) fragments suitable for fuzzy description logics, investigated by Straccia [28] and Hájek [14], and fuzzy logic programming, as studied e.g. by Vojtás [29]. Finally, we remark that validity for the axiomatizable sub-logic of $\forall$Ł characterized by linearly ordered MV-algebras is Σ_1-complete [8, 13].

The aim of this paper is to provide a proof-theoretic basis for first-order Łukasiewicz Logic as a starting point both for applications and deeper algorithmic investigations. We begin with a simple topological proof of the fact that while the usual Herbrand theorem does not hold for $\forall$Ł, an "approximate Herbrand theorem" (proved in a more complicated fashion by Novák in [24]) can be obtained instead. Essentially, for any valid existential formula, there exist Herbrand disjunctions for successive approximations to validity: for any $r < 1$ a disjunction exists that always takes a value greater than r. Following a similar proof for first-order Gödel Logic [2], we then use this result to give a proof-theoretic proof that first-order Łukasiewicz Logic admits Skolemization. The proof-theoretic treatment, unlike semantic proofs (obtained e.g. in the wider setting of Continuous Model Theory in [9]), implies that any system of functions can be considered as Skolem functions obeying some constraints, such as commutativity. This in turn allows greater flexibility for replacing functions in formulas with quantified terms.

We also make use of the approximate Herbrand theorem and Skolemization to define proof systems in the framework of hypersequents, a generalization of Gentzen sequents consisting of a multiset – intuitively, a disjunction – of sequents. Hypersequents were introduced by Avron in [1] and have been used to provide proof systems for a wide range of (first-order) fuzzy logics [6, 3, 19, 11]. In particular, a cut-free calculus GŁ for propositional Łukasiewicz Logic was developed in [20] using a non-standard interpretation of hypersequents. Here we add hypersequent versions of the usual quantifier rules for Gentzen's LK and LJ (as used e.g. for first-order Gödel Logic in [6]) to GŁ. We obtain a calculus G$\forall$Ł that, when extended with a cut rule, is complete with respect to linearly ordered MV-algebras. However, as we show with a suitable counterexample, cut-elimination does not hold for this extended calculus. On the other hand, adding an infinitary rule to G$\forall$Ł gives a cut-free system that is complete for the full logic $\forall$Ł. Finally, a cut-free calculus with finitary rules is obtained for the one-variable fragment by relaxing the eigenvariable condition in the quantifier rules.[4]

2 First-Order Łukasiewicz Logic

We will make use here of a usual first-order language. We have countable sets of predicate symbols, (object) constants, function symbols (with positive arity), and variables;

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[4] An earlier version of this paper has appeared as [4].

quantifiers $\forall$ and $\exists$; a binary connective $\rightarrow$; and a (logical) constant $\bot$, where:

$$\begin{array}{rlrl} \neg A & =_{def} A \rightarrow \bot & \top & =_{def} \neg\bot \\ A \oplus B & =_{def} \neg A \rightarrow B & A \odot B & =_{def} \neg(\neg A \oplus \neg B) \\ A \vee B & =_{def} (A \rightarrow B) \rightarrow B & A \wedge B & =_{def} \neg(\neg A \vee \neg B) \end{array}$$

It will also be helpful to define (with $n \in \mathbb{N}$):

$$\begin{array}{rlrl} 0.A & =_{def} \bot & (n+1).A & =_{def} A \oplus n.A \\ A^0 & =_{def} \top & A^{n+1} & =_{def} A \odot A^n \end{array}$$

In fact, for first-order Łukasiewicz Logic, as in the classical case, it is possible to define $(\exists x)A(x)$ as $\neg(\forall x)\neg A(x)$, but we will find it more convenient here to treat both quantifiers as primitive.

Terms t, s and formulas A, B, C are built inductively from the elements of this language in the usual manner, adopting standard notions of subformulas, scope, and distinguished bound variables x, y, z and free variables a, b. *Function-free*, *quantifier-free*, and *one-variable* formulas are formulas built using no function symbols, no quantifiers, and just one bound variable, respectively.[5] Also, we will call quantifier-free formulas containing no variables, *propositional*.

A sequence of terms $t_1, \ldots, t_n$ (where n may be 0, denoting the empty sequence) is often written $\bar{t}$. We denote a formula A with distinguished free variables among $\bar{x}$ by $A(\bar{x})$, and A with each free occurrence of x_i replaced by t_i for $i = 1 \ldots n$ by $A(\bar{t})$. We also write $(Q\bar{x})$ for the (possibly empty) sequence $(Qx_1) \ldots (Qx_n)$ where $Q \in \{\forall, \exists\}$.

Finite multisets of formulas, written directly as $[A_1, \ldots, A_n]$ where $[]$ is the empty multiset, are denoted $\mathit{\Gamma}, \mathit{\Delta}$. We write $\mathit{\Gamma}_1 \uplus \mathit{\Gamma}_2$ for the multiset sum of $\mathit{\Gamma}_1$ and $\mathit{\Gamma}_2$ and use $\mathit{\Gamma} \subseteq \mathit{\Delta}$ to indicate that $\mathit{\Delta}$ is a sub-multiset of $\mathit{\Gamma}$, and multiset-builder notation $[\,:\,]$ to construct multisets satisfying a particular property. We also let $\mathit{\Gamma}^n$ stand for the multiset sum of n copies of $\mathit{\Gamma}$ (in particular, $[A]^n$ is the multiset containing n copies of A), and $\star[A_1, \ldots, A_n]$ for the formula $A_1 \star \ldots \star A_n$ for $\star \in \{\wedge, \vee, \odot, \oplus\}$.

$\forall$Ł-*interpretations* $\mathcal{I} = (\mathcal{D}, v_{\mathcal{I}})$ consist of a non-empty set $\mathcal{D}$ and a valuation $v_{\mathcal{I}}$ that maps constants and object variables to elements of $\mathcal{D}$; n-ary function symbols to functions from $\mathcal{D}^n$ into $\mathcal{D}$; and n-ary predicate symbols to functions from $\mathcal{D}^n$ into $[0, 1]$. As usual, $v_{\mathcal{I}}$ is extended to all terms inductively by the condition $v_{\mathcal{I}}(f(t_1, \ldots, t_n)) = v_{\mathcal{I}}(f)(v_{\mathcal{I}}(t_1), \ldots, v_{\mathcal{I}}(t_n))$ for any n-ary function symbol f and terms $t_1, \ldots, t_n$. For an n-ary predicate symbol p and terms $t_1, \ldots, t_n$:

$$v_{\mathcal{I}}(p(t_1, \ldots, t_n)) = v_{\mathcal{I}}(p)(v_{\mathcal{I}}(t_1), \ldots, v_{\mathcal{I}}(t_n))$$

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[5] One-variable formulas usually allow the given variable to have both free and bound occurrences. This is not permitted here due to the distinction between free and bound variables, but has no impact on the decidability or complexity of the fragment.

For a variable x and element $d \in \mathcal{D}$, let $v_{\mathcal{I}}[x \leftarrow d]$ be the valuation obtained from $v_{\mathcal{I}}$ by changing $v_{\mathcal{I}}(x)$ to d. Then $v_{\mathcal{I}}$ is extended to all formulas by:

$$\begin{aligned} v_{\mathcal{I}}(\bot) &= 0 \\ v_{\mathcal{I}}(A \to B) &= \min(1, 1 - v_{\mathcal{I}}(A) + v_{\mathcal{I}}(B)) \\ v_{\mathcal{I}}((\forall x)A(x)) &= \inf\{v_{\mathcal{I}}[x \leftarrow a](A(x)) : a \in \mathcal{D}\} \\ v_{\mathcal{I}}((\exists x)A(x)) &= \sup\{v_{\mathcal{I}}[x \leftarrow a](A(x)) : a \in \mathcal{D}\} \end{aligned}$$

$\mathcal{I}$ *satisfies* a formula A if $v_{\mathcal{I}}(A) = 1$, and A is $\forall$Ł-*valid*, written $\models_{Ł} A$, if A is satisfied by all $\forall$Ł-interpretations. Two formulas A and B are $\forall$Ł-*equivalent*, written $A \sim_{Ł} B$, if $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(B)$ for all $\forall$Ł-interpretations $v_{\mathcal{I}}$.

It will also be crucial in this paper to consider the following notion of "approximate validity" for $\rhd \in \{>, \geq\}$ and $r \in [0, 1]$:

$$\models_{Ł}^{\rhd r} A \ \text{ iff } \ v_{\mathcal{I}}(A) \rhd r \text{ for all } \forall\text{Ł-interpretations } \mathcal{I}.$$

While the problem of checking validity of formulas, or approximate validity when $\rhd$ is $\geq$, is Π_2-complete [25], checking approximate validity when $\rhd$ is $>$ and r is rational is Σ_1-complete [7]. For one-variable formulas (i.e. the one-variable fragment), checking validity is decidable [26]. For the monadic fragment, consisting of formulas with predicate symbols of arity at most one, checking satisfiability is known to be Π_1-complete [25], but the complexity and indeed decidability of checking validity is an open problem. Finally, checking the validity of propositional formulas is known to be decidable, indeed co-NP complete [23].

Interpretations and validity can be generalized to a wider class of algebraic structures. An *MV-algebra* is an algebra $\mathbf{A} = \langle L, \oplus, \neg, 0\rangle$ such that:

1. $\langle L, \oplus, 0\rangle$ is a commutative monoid.
2. $\neg\neg x = x$.
3. $x \oplus \neg 0 = \neg 0$.
4. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

with $x \leq y$ defined as $\neg x \oplus y = \neg 0$, $1 =_{def} \neg 0$, and:

$$\begin{aligned} x \odot y &=_{def} \neg(\neg x \oplus \neg y) & \qquad x \to y &=_{def} \neg x \oplus y \\ x \vee y &=_{def} \neg(\neg x \oplus y) \oplus y & \qquad x \wedge y &=_{def} \neg(\neg(x \oplus \neg y) \oplus \neg y) \end{aligned}$$

An *MV-chain* is a linearly ordered MV-algebra; i.e. satisfying $x \leq y$ or $y \leq x$ for all $x, y \in L$. The most important example of an MV-chain is $\langle [0, 1], \oplus, \neg, 0\rangle$ where $x \oplus y = \min(1, x + y)$ and $\neg x = 1 - x$. In this case $x \to y = \min(1, 1 - x + y)$, $x \odot y = \max(0, x + y - 1)$, $x \wedge y = \min(x, y)$, and $x \vee y = \max(x, y)$.

Let $\mathbf{A} = \langle L, \oplus, \neg, 0\rangle$ be an MV-algebra. Then $\mathbf{A}$-*interpretations* $\mathcal{I} = (\mathcal{D}, v_{\mathcal{I}})$ are defined as for $\forall$Ł-interpretations except that n-ary predicate symbols are mapped by $v_{\mathcal{I}}$

to functions from $\mathcal{D}^n$ into L, and $v_\mathcal{I}$ is extended to all formulas by:

$$v_\mathcal{I}(\bot) = 0$$

$$v_\mathcal{I}(A \to B) = \begin{cases} v_\mathcal{I}(A) \to v_\mathcal{I}(B) & \text{if } v_\mathcal{I}(A) \text{ and } v_\mathcal{I}(B) \text{ are defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$v_\mathcal{I}((\forall x)A(x)) = \begin{cases} \inf\{v_\mathcal{I}[x \leftarrow d](A(x)) : d \in \mathcal{D}\} & \text{if } v_\mathcal{I}[x \leftarrow d](A(x)) \text{ for } d \in \mathcal{D} \\ & \text{and the infimum all exist} \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$v_\mathcal{I}((\exists x)A(x)) = \begin{cases} \sup\{v_\mathcal{I}[x \leftarrow d](A(x)) : d \in \mathcal{D}\} & \text{if } v_\mathcal{I}[x \leftarrow d](A(x)) \text{ for } d \in \mathcal{D} \\ & \text{and the supremum all exist} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Notice that if the required infimum or supremum does not exist in L, then the values of the corresponding universal or existential formulas, and all those in which they occur as subformulas, are undefined. The standard practice for dealing with such situations is simply to stipulate that they do not occur. That is, for each MV-algebra $\mathbf{A}$, we treat only *safe* $\mathbf{A}$-interpretations $v_\mathcal{I}$ where $v_\mathcal{I}(A)$ is defined for all formulas A. We then say, as in the earlier definition, that $\mathcal{I}$ *satisfies* a formula A if $v_\mathcal{I}(A) = 1$, and that A is $\mathbf{A}$-*valid* if A is satisfied by all safe $\mathbf{A}$-interpretations. Clearly, interpretations for the standard MV-chain $\langle [0,1], \oplus, \neg, 0\rangle$ mentioned above are safe, and indeed, as is trivial to check, validity in this algebra coincides with ∀Ł-validity.

MV-algebras arise as the algebraic semantics of propositional Łukasiewicz Logic. That is, for each propositional formula A:

$$\models_{\text{Ł}} A \quad \text{iff} \quad A \text{ is } \mathbf{A}\text{-valid for every MV-algebra } \mathbf{A}$$

We refer to [12] for details. Note however that at the first-order level, not only do we restrict our attention to MV-chains and safe interpretations, but also just the right-to-left direction holds in the above equivalence.

In this paper we will make brief use of a fundamental result for MV-algebras: the categorical equivalence of the category of MV-algebras with the category of abelian ℓ-groups with strong unit. Recall that an abelian ℓ-group is an algebra $\mathbf{G} = \langle L, \wedge, \vee, +, -, 0\rangle$ such that $\langle L, \wedge, \vee\rangle$ is a lattice, $\langle L, +, -, 0\rangle$ is an abelian group, and $+$ is order preserving; i.e. $x \leq y$ implies $z + x \leq z + y$ for all $x, y, z \in L$. Making use of the additive notation, we write $x - y$ for $-y + x$, nx for $x + \ldots + x$ (n times), and $\sum \Gamma$ for the sum $x_1 + \ldots + x_n$ in $\mathbf{G}$ of a multiset of elements $\Gamma = [x_1, \ldots, x_n]$ where $\sum[] = 0$. A strong unit for $\mathbf{G}$ is a member u of L such that for any $x \in L$, $nu \geq x$ for some $n \in \mathbb{N}$. Then more precisely, what we need is the following lemma:

Lemma 1 ([22]). *For every MV-chain* $\mathbf{A}$*, there is a linearly ordered abelian ℓ-group* $\mathbf{G} = \langle L, \wedge, \vee, +, -, 0\rangle$ *with strong unit* u, $\Xi(\mathbf{A}) = (\mathbf{G}, u)$*, such that* $\mathbf{A}$ *is isomorphic to* $\langle [0,u], \oplus, \neg, 0\rangle$ *where* $[0,u] = \{x \in L : 0 \leq x \leq u\}$, $x \oplus y = u \wedge (x + y)$*, and* $\neg x = u - x$*, and the* sup*s and* inf*s of* $\mathbf{A}$ *coincide with the corresponding* sup*s and* inf*s of* $\langle [0,u], \oplus, \neg, 0\rangle$.

For convenience, we can assume that $\mathbf{A}$ just *is* the algebra $\langle [0,u], \oplus, \neg, 0\rangle$ of $\Xi(\mathbf{A})$.

(Ł1) $A \to (B \to A)$
(Ł2) $(A \to B) \to ((B \to C) \to (A \to C))$
(Ł3) $((A \to B) \to B) \to ((B \to A) \to A)$
(Ł4) $((A \to \bot) \to (B \to \bot)) \to (B \to A)$
(∀1) $(\forall x)A(x) \to A(t)$
(∀2) $(\forall x)(A \to B) \to (A \to (\forall x)B)$ where x is not free in A

$$\frac{A \quad A \to B}{B}\ (mp) \qquad\qquad \frac{A}{(\forall x)A}\ (gen)$$

Fig. 1: The Hilbert System H∀Ł

Finally, we note that the axiomatization H∀Ł in Figure 1, introduced by Hájek in [13] (simplifying previous axiomatizations of Hay [15] and Belluce and Chang [8, 7]), corresponds to validity in MV-chains, where formulas are defined without distinguishing between free and bound variables, and $\exists A =_{def} \neg\forall\neg A$.

Theorem 1 ([13]). $\vdash_{\mathsf{H\forall Ł}}$ *A iff A is* $\mathbf{A}$*-valid for all MV-chains* $\mathbf{A}$.

3 An Approximate Herbrand Theorem

Our aim in this section will be to show the failure of the Herbrand theorem for first-order Łukasiewicz Logic, compensated for somewhat by the success instead of an “approximate” Herbrand theorem. Before moving on to the deficiencies of ∀Ł, however, let us consider one of its more attractive features. Like Classical Logic, ∀Ł has the full quota of “quantifier shifts”. That is, we have the following ∀Ł-equivalences, where by definition (the bound variable) x does not occur free in A:

$$A \to (\forall x)B \ \sim_{Ł} (\forall x)(A \to B) \qquad (\forall x)B \to A \ \sim_{Ł} (\exists x)(B \to A)$$
$$A \to (\exists x)B \ \sim_{Ł} (\exists x)(A \to B) \qquad (\exists x)B \to A \ \sim_{Ł} (\forall x)(B \to A))$$

Let us write $(Q\bar{x})A(\bar{x})$ for a formula $(Q_1x_1)\dots(Q_nx_n)A(x_1,\dots,x_n)$ where $Q_i \in \{\forall, \exists\}$ for $i = 1\dots n$. A *prenex formula* is a formula $(Q\bar{x})P(\bar{x})$ where P is quantifier-free. Then in ∀Ł as in Classical Logic, given any formula A, we can rewrite all bound variables to new variables and use the above equivalences (left-to-right) as rewrite rules to push all quantifiers to the outside.

Theorem 2. *Any formula is* ∀Ł*-equivalent to a prenex formula.*

Let us now recall some basic notions relating to Herbrand's theorem. Let A be a formula, and let $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{P}$ be the constants, function symbols, and predicate symbols occurring in A, respectively, adding a constant if the first is empty. The *Herbrand universe* $U(A)$ of A is the set of ground (i.e. containing no variables) terms built using $\mathcal{C}$ and $\mathcal{F}$. I.e. $U(A) = \bigcup_{n=0}^{\infty} U_n(A)$ where:

$$U_0(A) = \mathcal{C}$$
$$U_{n+1}(A) = U_n(A) \cup \{f(t_1,\dots,t_k) : t_1,\dots,t_k \in U_n(A) \text{ and } f \in \mathcal{F} \text{ with arity } k\}$$

The *Herbrand base* $B(A)$ of A is defined as:

$$B(A) = \{p(t_1, \ldots, t_k) : t_1, \ldots, t_k \in U(A) \text{ and } p \in \mathcal{P} \text{ with arity } k\}$$

For a logic L, the usual Herbrand Theorem for existential formulas states that a formula $(\exists \bar{x})P(\bar{x})$, where P is quantifier-free, is L-valid iff a disjunction $\bigvee_{i=1}^{n} P(\bar{t}_i)$ is L-valid for some $\bar{t}_1, \ldots, \bar{t}_n \in U(P)$. However, as we will now show, this formulation does not hold for $\forall$Ł. First, observe that $\models_{Ł} (\exists x)p(x) \to (\exists y)p(y)$ and hence, using the quantifier-shifting equivalences above:

$$\models_{Ł} (\exists y)(\forall x)(p(x) \to p(y))$$

It then follows by a simple semantic argument that:

$$\models_{Ł} (\exists y)(p(f(y)) \to p(y))$$

So if Herbrand's theorem holds for $\forall$Ł, then for some constant c and $n \geq 1$:

$$\models_{Ł} \bigvee_{i=1}^{n} (p(f^i(c)) \to p(f^{i-1}(c)))$$

where $f^0(c) = c$ and $f^{i+1}(c) = f(f^i(c))$ for $i \in \mathbb{N}^+$. But now define $v_{\mathcal{I}}(p(f^i(c))) = i/n$ for $n = 0, 1, \ldots, n$. It follows that:

$$v_{\mathcal{I}}(p(f^i(c))) > v_{\mathcal{I}}(p(f^{i-1}(c))) \quad \text{for } i = 1 \ldots n$$

Hence also $v_{\mathcal{I}}(\bigvee_{i=1}^{n}(p(f^i(c)) \to p(f^{i-1}(c))) < 1$. But this contradicts the validity of the formula, so the Herbrand theorem must fail.

Take another look at this formula $\bigvee_{i=1}^{n}(p(f^i(c)) \to p(f^{i-1}(c)))$, however. Although this is not $\forall$Ł-valid, it comes within "one nth" of being so. Just observe that for any $a_0, a_1, \ldots, a_n \in [0, 1]$:

$$\min\{a_{i-1} - a_i : 1 \leq i \leq n\} \leq 1/n$$

It follows easily that for any $r < 1 - 1/n$:

$$\models_{Ł}^{>r} \bigvee_{i=1}^{n} (p(f^i(c)) \to p(f^{i-1}(c)))$$

That is, we can characterize successive "Herbrand approximations" to $(\exists y)(p(f(y)) \to p(y))$ that come arbitrarily close to 1. This illustrates a more general phenomenon, captured by the following approximate Herbrand theorem:

Theorem 3. $\models_{Ł} (\exists \bar{x})P(\bar{x})$ *where* P *is quantifier-free iff for all* $r < 1$*:*

$$\models_{Ł}^{>r} \bigvee_{i=1}^{n} P(\bar{t}_i) \text{ for some } \bar{t}_1, \ldots, \bar{t}_n \in U(P)$$

Proof. We refer to [30] for all topological terminology. Suppose that $\models_{\text{Ł}}^{>r} \bigvee_{i=1}^{n} P(\bar{t}_i)$ for all $r < 1$. Then $\models_{\text{Ł}}^{>r} (\exists \bar{x})P(\bar{x})$ for all $r < 1$, so clearly $\models_{\text{Ł}} (\exists \bar{x})P(\bar{x})$. For the other direction, fix $r < 1$. Notice that any mapping from $B(P)$ into $[0,1]$ – a member either of $[0,1]^k$ for some k if $B(P)$ is finite, or of the Hilbert cube $[0,1]^\omega$ if $B(P)$ is countably infinite – may be viewed as a valuation $v_{\mathcal{I}}$ for an interpretation $\mathcal{I}$. In either case ($[0,1]^\omega$ using the Tychonoff Theorem), $[0,1]^{B(P)}$ is a compact space with respect to the product topology. Now for each $\bar{t} \in U(P)$ define:

$$S(\bar{t}) = \{v_{\mathcal{I}} \in [0,1]^{B(P)} : v_{\mathcal{I}}(P(\bar{t})) \leq r\}$$

Since P is quantifier-free and the propositional connectives $\rightarrow$ and $\bot$ are interpreted by continuous functions on $[0,1]$, each $S(\bar{t})$ is a closed subset of $[0,1]^{B(P)}$. Consider:

$$S = \{S(\bar{t}) : \bar{t} \in U(P)\}$$

We have two possibilities:

1. Suppose that for some $\{S(\bar{t}_1), \ldots, S(\bar{t}_n)\} \subseteq S$:

   $$\bigcap_{i=1} S(\bar{t}_i) = \emptyset$$

   Then for every interpretation $\mathcal{I}$, $v_{\mathcal{I}}(P(\bar{t}_i)) > r$ for some $i \in \{1, \ldots, n\}$. I.e. $\models_{\text{Ł}}^{>r} \bigvee_{i=1}^{n} P(\bar{t}_i)$ as required.

2. Otherwise, every finite subset of S has a non-empty intersection. Since S is a collection of closed subsets of $[0,1]^{B(P)}$, by the finite intersection property for compact spaces, S also has a non-empty intersection. I.e., there exists $v_{\mathcal{I}}$ such that $v_{\mathcal{I}}(P(\bar{t})) \leq r$ for all $\bar{t} \in U(P)$. So $v_{\mathcal{I}}((\exists \bar{x})P(\bar{x})) \leq r$, a contradiction. □

This approximate Herbrand theorem has a nice corollary. Let $F = (\forall \bar{x})(\exists \bar{y})P(\bar{x}, \bar{y})$ where P is both quantifier-free and function-free. Then $\models_{\text{Ł}} F$ iff $\models_{\text{Ł}} (\exists \bar{y})P(\bar{c}, \bar{y})$ for some new constants $\bar{c}$. Let $\mathcal{C}$ be the (finite) set of constants occurring in $(\exists \bar{y})P(\bar{c}, \bar{y})$, adding one if the set is empty. Using the previous theorem:

$$\models_{\text{Ł}} F \quad \text{iff} \quad \text{for all } r < 1, \ \models_{\text{Ł}}^{>r} \bigvee_{i=1}^{n} P(\bar{c}, \bar{t}_i) \text{ for some } \bar{t}_1, \ldots, \bar{t}_n \in \mathcal{C} \quad \text{iff} \quad \models_{\text{Ł}} \bigvee_{\bar{d} \in \mathcal{C}} P(\bar{c}, \bar{d})$$

But checking validity in propositional Łukasiewicz Logic is decidable, so we have established the following:

Proposition 1. *The function-free* $\forall\exists$*-fragment of* $\forall$Ł *is decidable.*

Notice also that for any formula A in the function-free one-variable fragment of $\forall$Ł, we can find an existential function-free formula $(\exists \bar{x})P(\bar{x})$ such that $\models_{\forall\text{Ł}} A$ iff $\models_{\forall\text{Ł}} (\exists \bar{x})P(\bar{x})$. Consider the following translations A^+ and A^-, assuming harmlessly that the ith occurrence of a quantifier Q in A is annotated as Q^i and each a_i is a variable not occurring in A:

$$\begin{array}{rlrl}
p(\bar{x})^+ &= p(\bar{x}) & p(\bar{x})^- &= p(\bar{x}) \\
\bot^+ &= \bot & \bot^- &= \bot \\
(B \to C)^+ &= B^- \to C^+ & (B \to C)^- &= B^+ \to C^- \\
((\forall^i x)B(x))^+ &= B(a_i)^+ & ((\forall x)^i B(x))^- &= (\forall x)B(x)^- \\
((\exists^i x)B(x))^+ &= (\exists x)B(x)^+ & ((\exists x)^i B(x))^- &= B(a_i)^+
\end{array}$$

A is a one-variable formula, so in each subformula $(\forall x)B(x)$ or $(\exists x)B(x)$ of A, no variable is bound by another quantifier. Hence using the continuity of the connectives in $\forall$Ł (pushing sups and infs outwards), it follows easily that $\models_{Ł} A$ iff $\models_{Ł} A^+$. But using $\forall$Ł-equivalences to push out the remaining quantifiers, A^+ is equivalent to an existential function-free formula. Hence, by the previous proposition:

Corollary 1. *The function-free one-variable fragment of* $\forall$Ł *is decidable.*

4 Skolemization

We will now use the approximate Herbrand theorem to provide a proof-theoretic proof of Skolemization for $\forall$Ł. For Classical Logic, Skolemization usually involves removing existential quantifiers and preserving satisfiability. Here we follow common terminology for fuzzy logics (see e.g. [2]) and remove universal quantifiers, hoping rather to preserve validity.

Let A be a prenex formula and assume harmlessly that the ith occurrence of $\forall$ is labelled $\forall^i$ and that no function symbol or constant f_i occurs in A. Then the *Skolem form* A^S of A is defined by induction as follows:

(1) If A is of the form $(\exists\bar{x})P(\bar{x})$ where P is quantifier-free, then A^S is $(\exists\bar{x})P(\bar{x})$.
(2) If A is of the form $(\exists\bar{x})(\forall^i y)B(\bar{x}, y)$, then A^S is $((\exists\bar{x})B(\bar{x}, f_i(\bar{x}))^S$.

Our aim is to prove that $\models_{Ł} A$ iff $\models_{Ł} A^S$ for any prenex formula A. The first step towards establishing the difficult right-to-left direction of this equivalence is to show that if a Herbrand disjunction for the Skolem form of a prenex formula is $\geq r$ valid, then the formula is itself valid to the same degree.

Lemma 2. *Let* $(\exists\bar{x})P^F(\bar{x})$ *be the Skolem form of* $(Q\bar{y})P(\bar{y})$. *Then* $[(Q\bar{y})P(\bar{y})]$ *is derivable from any finite non-empty sub-multiset of* $[P^F(\bar{t}) : \bar{t} \in U(P^F)]$ *using the rules:*

$$\frac{\Gamma \uplus [A(t)]}{\Gamma \uplus [(\forall x)A(x)]} \qquad \frac{\Gamma \uplus [A(s)]}{\Gamma \uplus [(\exists x)A(x)]} \qquad \frac{\Gamma \uplus [A, A]}{\Gamma \uplus [A]} \qquad \frac{\Gamma}{\Gamma \uplus [A]}$$

where in the leftmost rule, t is any ground term not occurring in Γ or A.

Proof. Let us assume that each occurrence of $\forall$ in $[(Q\bar{y})P(\bar{y})]$ is labelled $f(z_1, \ldots, z_n)$ where f is the constant or n-ary function symbol in $(\exists\bar{x})P^F(\bar{x})$ introduced by Skolemization for this occurrence of $\forall$, and $z_1, \ldots, z_n$ are the existentially bound variables preceding the occurrence in $(Q\bar{y})P(\bar{y})$. More generally, we will allow this occurrence of $\forall$ to be labelled $f(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n$ are terms. We will also suppose that substituting for a variable in such labelled formulas extends to substituting also in the labels. In particular, given a labelled formula $A(x)$, $A(t)$ is obtained by replacing all free occurrences of x in $A(x)$ by t, including all those in the labels.

Now let $\Gamma_0 = [(Q\bar{y})P(\bar{y})]$. Given Γ_j, let Γ_{j+1} be the smallest multiset satisfying:

(1) $\Gamma_j \subseteq \Gamma_{j+1}$.
(2) If $(\forall x)B(x) \in \Gamma_{j+1}$ and $f(\bar{t})$ labels $\forall$, then $B(f(\bar{t})) \in \Gamma_{j+1}$.
(3) If $(\exists x)B(x) \in \Gamma_{j+1}$, then $B(s) \in \Gamma_{j+1}$ for all $s \in U_j(P^F)$.

Notice first that each Γ_j can be derived from Γ_{j+1} using the given rules. The only difficulty could be that for (2), the term $f(\bar{t})$ occurs already in the conclusion. However, each occurrence of $\forall$ is labelled with a different constant or function symbol f with arguments determined uniquely by the terms chosen for the preceding occurrences of $\exists$. Hence the new formula in the premise for such a case must already occur in the conclusion. The desired result is then a consequence of the following:

Claim: if $\bar{t} \in U(P^F)$, then $P^F(\bar{t}) \in \Gamma_j$ for some $j \in \mathbb{N}$.

This is proved by an easy induction on the number of quantifiers in $(Q\bar{y})P(\bar{y})$. It implies that any finite non-empty sub-multiset of $[P^F(\bar{t}) : \bar{t} \in U(P^F)]$ is a sub-multiset of some Γ_j. But Γ_0 can be derived from Γ_j using the rules, and Γ_j can be derived from any sub-multiset of Γ_j using the rightmost rule, so we are done. $\square$

Proposition 2. *Let $(\exists\bar{x})P^F(\bar{x})$ be the Skolem form of $(Q\bar{y})P(\bar{y})$. For any $r \in [0,1]$:*

$$\text{if } \models_{Ł}^{\geq r} \bigvee_{i=1}^{n} P^F(\bar{t}_i) \text{ for some } \bar{t}_1, \ldots, \bar{t}_n \in U(P^F), \text{ then } \models_{Ł}^{\geq r} (Q\bar{y})P(\bar{y}).$$

Proof. For each rule in Lemma 2, with premise Γ and conclusion Δ, it is easy to see that $\models_{Ł}^{\geq r} \bigvee \Gamma$ implies $\models_{Ł}^{\geq r} \bigvee \Delta$. So the result follows by a simple induction on the height of a derivation using these rules of $[(Q\bar{y})P(\bar{y})]$ from $[P^F(\bar{t}_1), \ldots, P^F(\bar{t}_n)]$. $\square$

We now establish Skolemization for the prenex formulas of $\forall$Ł by combining this last result with the approximate Herbrand theorem.

Theorem 4. *Let $(\exists\bar{x})P^F(\bar{x})$ be the Skolem form of $(Q\bar{y})P(\bar{y})$. Then:*

$$\models_{Ł} (\exists\bar{x})P^F(\bar{x}) \quad \textit{iff} \quad \models_{Ł} (Q\bar{y})P(\bar{y})$$

Proof. The right-to-left direction follows easily using standard quantifier properties of $\forall$Ł. For the other direction, suppose that $\models_{Ł} (\exists\bar{x})P^F(\bar{x})$. By Theorem 3, for all $r < 1$, there exist tuples of terms, $\bar{t}_1, \ldots, \bar{t}_n$, in $U(P^F)$ such that $\models_{Ł}^{\geq r} \bigvee_{i=1}^{n} P^F(\bar{t}_i)$. But then by Proposition 2, for all $r < 1$, $\models_{Ł}^{\geq r} (Q\bar{y})P(\bar{y})$. Hence $\models_{Ł} (Q\bar{y})P(\bar{y})$ as required. $\square$

Skolemization allows us to extend the approximate Herbrand theorem to the whole of $\forall$Ł. We just put the formula into prenex form, use Theorem 4 to find an appropriate existential formula, and apply Theorem 3.

Corollary 2. *Let A be a formula and let $(\exists\bar{x})P^F(\bar{x})$ be the Skolem form of an equivalent prenex formula for A. Then $\models_{Ł} A$ iff $\models_{Ł} (\exists\bar{x})P^F(\bar{x})$ iff for all $r < 1$:*

$$\models_{Ł}^{>r} \bigvee_{i=1}^{n} P^F(\bar{t}_i) \text{ for some } \bar{t}_1, \ldots, \bar{t}_n \in U(P^F)$$

Moreover, we can use this approximate Herbrand theorem to sketch an alternative proof of a completeness result for the Hilbert system H∀Ł provided in [13] (following [15, 8, 7]). First notice that for any formula B and $k \in \mathbb{N}^+$:

$$\begin{array}{rl} 1 - 1/k \leq v_{\mathcal{I}}(B) & \text{iff } v_{\mathcal{I}}(\neg B) \leq 1/k \\ & \text{iff } kv_{\mathcal{I}}(\neg B) \leq 1 \\ & \text{iff } (k-1)v_{\mathcal{I}}(\neg B) \leq v_{\mathcal{I}}(B) \\ & \text{iff } 1 \leq v_{\mathcal{I}}(((k-1).\neg B) \to B) \\ & \text{iff } 1 \leq v_{\mathcal{I}}(B \oplus B^{k-1}) \end{array}$$

Now for any formula A, let $(\exists\bar{x})P^F(\bar{x})$ be the Skolem form of a prenex formula equivalent to A. We note without proof that A is H∀Ł-derivable from $(\exists\bar{x})P^F(\bar{x})$. Moreover, if $\models_{Ł} A$, then by the approximate Herbrand theorem, for all $k \in \mathbb{N}^+$:

$$\models_{Ł}^{\geq 1-1/k} \bigvee_{i=1}^{n} P^F(\bar{t}_i) \text{ for some } \bar{t}_1, \ldots, \bar{t}_n \in U(P^F)$$

So by the above reasoning and the propositional completeness of H∀Ł, for all $k \in \mathbb{N}^+$:

$$\vdash_{\mathsf{H}\forall Ł} B \oplus B^{k-1} \quad \text{where } B = \bigvee_{i=1}^{n} P^F(\bar{t}_i) \text{ for some } \bar{t}_1, \ldots, \bar{t}_n \in U(P^F)$$

But $(\exists\bar{x})P^F(\bar{x})$ is H∀Ł-derivable from any such B. So for all $k \in \mathbb{N}^+$, $A \oplus A^{k-1}$ is H∀Ł-derivable from $B \oplus B^{k-1}$, and we have:

Theorem 5 ([13]). $\models_{Ł} A$ *iff* $\vdash_{\mathsf{H}\forall Ł} A \oplus A^k$ *for all* $k \in \mathbb{N}^+$.

We remark finally that many of the results described in this section and the previous one can be generalized to a wide class of first-order fuzzy logics based on continuous t-norms. In particular, although it can be shown that Herbrand's theorem holds only for Gödel Logic in this family, the approximate Herbrand theorem, Skolemization, and decidability results, all hold given certain quite general conditions on the logic [5].

5 The Hypersequent Calculus GŁ

We will define proof systems for Łukasiewicz Logic in the framework of *hypersequents*; finite multisets of sequents, written:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

where Γ_i and Δ_i are finite multisets of formulas for $i = 1 \ldots n$.

Validity is extended to hypersequents as follows:

Initial Sequents

$$\frac{}{A \Rightarrow A}\ (id) \qquad\qquad \frac{}{\Rightarrow}\ (\Lambda) \qquad\qquad \frac{}{\bot \Rightarrow A}\ (\bot\Rightarrow)$$

Structural Rules:

$$\frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (ew) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (ec) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}\ (wl)$$

$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (split) \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (mix)$$

Logical Rules

$$\frac{\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta}\ (\to\Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta}\ (\Rightarrow\to)$$

Fig. 2: The Hypersequent Calculus GŁ

Definition 1. *Let* $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ *be a hypersequent. Then* $\mathcal{G}$ *is* ∀Ł*-valid, written* $\models_{\text{Ł}} \mathcal{G}$*, if for all* ∀Ł*-interpretations* $\mathcal{I}$*:*

$$\sum [v_{\mathcal{I}}(A) - 1 : A \in \Gamma_i] \leq \sum [v_{\mathcal{I}}(B) - 1 : B \in \Delta_i] \text{ for some } i \in \{1, \ldots, n\}.$$

More generally, for an MV-chain $\mathbf{A}$*,* $\mathcal{G}$ *is* $\mathbf{A}$*-valid if for all safe* $\mathbf{A}$*-interpretations* $\mathcal{I}$*:*

$$\sum [v_{\mathcal{I}}(A) - u : A \in \Gamma_i] \leq \sum [v_{\mathcal{I}}(B) - u : B \in \Delta_i] \text{ for some } i \in \{1, \ldots, n\}$$

in $\Xi(\mathbf{A}) = (G, u)$*, the abelian* ℓ*-group* $\mathbf{G} = \langle L, \wedge, \vee, +, -, 0 \rangle$ *with strong unit* u*.*

These definitions – where hypersequents are interpreted using sums of elements in abelian ℓ-groups rather than formulas of ∀Ł – are rather non-standard. In particular, multisets of formulas on the left and right of sequents are not interpreted using the operations of an MV-chain, but "outside the logic" using the addition $+$ of the corresponding abelian ℓ-group.[6] Nevertheless, note that for formulas we still retain the usual notion of validity, i.e.

$$\models_{\text{Ł}} A \text{ iff } \models_{\text{Ł}} \Rightarrow A$$

The hypersequent calculus GŁ for propositional Łukasiewicz Logic introduced in [20] is displayed in Figure 2. Note that Γ_1, Γ_2 stands for the multiset union $\Gamma_1 \uplus \Gamma_2$, and Γ, A for $\Gamma \uplus [A]$, and that $\mathcal{G}$ denotes an arbitrary "side-hypersequent" occurring in both the premises and conclusion of a rule.

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[6] We remark that in [20], this interpretation is represented as an embedding of Łukasiewicz Logic into Abelian Logic, the logic of abelian ℓ-groups.

GŁ is cut-free by definition and all the rules have the subformula property. Moreover, natural rules for the defined connectives can be obtained, e.g. for $\wedge$ and $\vee$:

$$\frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \wedge B \Rightarrow \Delta}\ (\wedge\Rightarrow)_1 \qquad \frac{\mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \wedge B \Rightarrow \Delta}\ (\wedge\Rightarrow)_2$$

$$\frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \vee B \Rightarrow \Delta}\ (\vee\Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow A, \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \wedge B, \Delta}\ (\Rightarrow\wedge)$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \vee B, \Delta}\ (\Rightarrow\vee)_1 \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \vee B, \Delta}\ (\Rightarrow\vee)_2$$

The standard version of the implication right rule is derivable when only one formula appears on the right (since the left premise $\mathcal{G} \mid \Gamma \Rightarrow$ in this case is derivable using (ew), (wl), and (Λ)):

$$\frac{\mathcal{G} \mid \Gamma, A \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow A \to B}\ (\Rightarrow\to)_1$$

Example 1. We illustrate this calculus with a derivation for $(Ł3)$ from H∀Ł:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,(id) \quad \dfrac{}{A \Rightarrow A}\,(id)}{B, A \Rightarrow A, B}\,(mix)}{B, B \to A \Rightarrow A}\,(\to\Rightarrow) \quad \dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,(id) \quad \dfrac{}{A \Rightarrow A}\,(id)}{B, A \Rightarrow A, B}\,(mix)}{B, B \to A, A \Rightarrow A, B}\,(wl)}{B, B \to A \Rightarrow A, A \to B}\,(\Rightarrow\to)}{(A \to B) \to B, B \to A \Rightarrow A}\,(\to\Rightarrow)}{\dfrac{(A \to B) \to B \Rightarrow (B \to A) \to A}{\Rightarrow ((A \to B) \to B) \to ((B \to A) \to A)}\,(\Rightarrow\to)_1}\,(\Rightarrow\to)_1$$

Hypersequents are not needed to prove this or indeed any of the other propositional axioms $(Ł1)$-$(Ł4)$ of H∀Ł. Nevertheless, they are essential to prove other ∀Ł-valid formulas such as $A \to (B \to ((A \to (A \to C)) \to ((B \to (B \to C)) \to C)))$.

Soundness and completeness proofs were presented for GŁ in [20], the latter a semantic proof making use of the invertibility of $(\Rightarrow\to)$ and a derived version of $(\to\Rightarrow)$.

Theorem 6 ([20]). *Let $\mathcal{G}$ be a propositional hypersequent. Then $\vdash_{\mathsf{G}Ł} \mathcal{G}$ iff $\models_{Ł} \mathcal{G}$.*

It follows easily from this theorem that the following cut rule and (inter-derivable) cancellation rule are admissible for GŁ:

$$\frac{\mathcal{G} \mid \Gamma_1, A \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow A, \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (cut) \qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (can)$$

Syntactic eliminations of these rules were provided in [10].

Theorem 7 ([10]). *Cut elimination holds for* GŁ + (cut) *and cancellation elimination holds for* GŁ + (can).

6 Adding Quantifiers

One of the most attractive features of sequent calculi such as Gentzen's LJ and LK for Intuitionistic Logic and Classical Logic, respectively, is that the first-order calculus is obtained by extending the propositional part with natural rules for the quantifiers. This feature is shared by hypersequent calculi for many fuzzy logics such as first-order Gödel Logic [6] and Monoidal t-norm Logic [3] where the rules added are just "hypersequent versions" of those for LJ and LK. In this section we consider the effect of adding these usual quantifier rules to GŁ, warning the reader in advance that progress to the first-order level will not be quite so smooth for this logic.

Definition 2. *Let* G∀Ł *be* GŁ *with the added rules:*

$$\dfrac{\mathcal{G} \mid \Gamma, A(t) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\forall x)A(x) \Rightarrow \Delta}\;(\forall\Rightarrow) \qquad\qquad \dfrac{\mathcal{G} \mid \Gamma \Rightarrow A(a), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\forall x)A(x), \Delta}\;(\Rightarrow\forall)$$

$$\dfrac{\mathcal{G} \mid \Gamma, A(a) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\exists x)A(x) \Rightarrow \Delta}\;(\exists\Rightarrow) \qquad\qquad \dfrac{\mathcal{G} \mid \Gamma \Rightarrow A(t), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\exists x)A(x), \Delta}\;(\Rightarrow\exists)$$

where the eigenvariable a does not occur in the conclusion of $(\Rightarrow\forall)$ or $(\exists\Rightarrow)$.

Example 2. Consider the following G∀Ł-derivation for $(\forall 2)$ from H∀Ł:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{B(a) \Rightarrow B(a)}\;(id) \quad \overline{A \Rightarrow A}\;(id)}{B(a), A \Rightarrow A, B(a)}\;(mix)}{A \to B(a), A \Rightarrow B(a)}\;(\to\Rightarrow)}{(\forall x)(A \to B), A \Rightarrow B(a)}\;(\forall\Rightarrow)}{(\forall x)(A \to B), A \Rightarrow (\forall x)B}\;(\Rightarrow\forall)}{(\forall x)(A \to B) \Rightarrow A \to (\forall x)B}\;(\Rightarrow\to)_1}{\Rightarrow (\forall x)(A \to B) \to (A \to (\forall x)B)}\;(\Rightarrow\to)_1$$

Notice that it is crucial here that x cannot occur free in A, since in that case $A(a)$ would appear on the left in the second application of (id), with $A(x)$ on the right.

Establishing soundness for G∀Ł with respect to the interpretation $\models_{Ł}$ is straightforward.

Theorem 8. *If $\vdash_{\mathsf{G\forall Ł}} \mathcal{H}$, then $\models_{Ł} \mathcal{H}$.*

Proof. By induction on the height of a derivation of $\mathcal{H}$ in G∀Ł. Since the soundness of the other rules is checked in [20], we can just consider the quantifier rules. Also note that we can ignore the side-hypersequent $\mathcal{G}$ occurring in both the premises and conclusion of the rules, since clearly if an inequality holds for a sequent of $\mathcal{G}$ in a premise, then it holds also for the same sequent in the conclusion. For $(\Rightarrow\forall)$, suppose that for every interpretation $\mathcal{I}$:

$$\sum[v_{\mathcal{I}}(C) - 1 : C \in \Gamma] \leq (v_{\mathcal{I}}(A(a)) - 1) + \sum[v_{\mathcal{I}}(D) - 1 : D \in \Delta]$$

where a does not occur in $\mathit{\Gamma}$, $\mathit{\Delta}$, or A. But now notice that for any interpretation $\mathcal{I}$ and $K \in \mathbb{R}$, if $K \leq v_{\mathcal{I}}[a \leftarrow d](A(a))$ for all $d \in \mathcal{D}$, then also $K \leq \inf\{v_{\mathcal{I}}[x \leftarrow d](A(x)) : d \in \mathcal{D}\} = v_{\mathcal{I}}((\forall x)A(x))$. Hence:

$$\sum[v_{\mathcal{I}}(C) - 1 : C \in \mathit{\Gamma}] \leq (v_{\mathcal{I}}((\forall x)A(x)) - 1) + \sum[v_{\mathcal{I}}(D) - 1 : D \in \mathit{\Delta}]$$

as required. For $(\forall\Rightarrow)$, suppose that for some term t:

$$\sum[v_{\mathcal{I}}(C) - 1 : C \in \mathit{\Gamma}] + (v_{\mathcal{I}}(A(t)) - 1) \leq \sum[v_{\mathcal{I}}(D) - 1 : D \in \mathit{\Delta}]$$

Then since $v_{\mathcal{I}}((\forall x)A(x)) \leq v_{\mathcal{I}}(A(t))$:

$$\sum[v_{\mathcal{I}}(C) - 1 : C \in \mathit{\Gamma}] + (v_{\mathcal{I}}((\forall x)A(x)) - 1) \leq \sum[v_{\mathcal{I}}(D) - 1 : D \in \mathit{\Delta}]$$

Cases for the existential quantifier rules are very similar. □

If we extend G∀Ł with (cut) or (can), then we obtain soundness and completeness with respect to MV-chains and the axiomatization H∀Ł (also, the previous theorem is obtained as a corollary). In fact, this is about as much as we could hope for, since ∀Ł is not recursively enumerable.

Theorem 9. *For any formula A, the following are equivalent:*

(1) $\vdash_{\mathsf{G\forall Ł}+(cut)} \Rightarrow A$.
(2) $\vdash_{\mathsf{H\forall Ł}} A$.
(3) *A is $\mathbf{A}$-valid for every MV-chain $\mathbf{A}$.*

Proof. The equivalence of (2) and (3) is Theorem 1. To show that (2) implies (1), we first observe that (gen) and (mp) are admissible for G∀Ł+(cut) using $(\Rightarrow\forall)$ and (cut), respectively. Also, the axioms (Ł1)-(Ł4) and (∀1)-(∀2) are G∀Ł-derivable. Hence the derivability of A in H∀Ł implies the derivability of $\Rightarrow A$ in G∀Ł + (cut). To show that (1) implies (3), we prove the following:

Claim: If $\vdash_{\mathsf{G\forall Ł}+(cut)} \mathcal{G}$, then $\mathcal{G}$ is $\mathbf{A}$-valid for every MV-chain $\mathbf{A}$.

We proceed by induction on the height of a derivation of $\mathcal{G}$ in G∀Ł+(cut). This requires checking the soundness of each rule. Recall that for each MV-chain $\mathbf{A}$ and abelian ℓ-group $\mathbf{G} = \langle L, \wedge, \vee, +, -, 0\rangle$ with strong unit u such that $\Xi(\mathbf{A}) = (\mathbf{G}, u)$, $\mathbf{A}$ is (isomorphic to) $\langle [0, u], \oplus, \neg, 0\rangle$ where $x \oplus y =_{def} u \wedge (x + y)$ and $\neg x =_{def} u - x$. The soundness proofs mimic almost exactly the particular case of the standard MV-chain on $[0, 1]$. In particular the soundness of the quantifier rules follows as in the proof of Theorem 8. As an example, consider $(\rightarrow\Rightarrow)$, again disregarding the side-hypersequent $\mathcal{G}$. Suppose that for some MV-chain $\mathbf{A}$ and safe $\mathbf{A}$-interpretation $\mathcal{I}$:

$$\sum[v_{\mathcal{I}}(C) - u : C \in \mathit{\Gamma}] + (v_{\mathcal{I}}(B) - u) \leq \sum[v_{\mathcal{I}}(D) - u : D \in \mathit{\Delta}] + (v_{\mathcal{I}}(A) - u)$$

Then using properties of addition and subtraction in $\Xi(\mathbf{A})$:

$$\sum[v_{\mathcal{I}}(C) - u : C \in \mathit{\Gamma}] + ((u - v_{\mathcal{I}}(A) + v_{\mathcal{I}}(B)) - u) \leq \sum[v_{\mathcal{I}}(D) - u : D \in \mathit{\Delta}]$$

Hence, since $v_{\mathcal{I}}(A \to B) = \neg v_{\mathcal{I}}(A) \oplus v_{\mathcal{I}}(B) = u \wedge (u - v_{\mathcal{I}}(A) + v_{\mathcal{I}}(B))$:

$$\sum[v_{\mathcal{I}}(C) - u : C \in \Gamma] + (v_{\mathcal{I}}(A \to B) - u) \leq \sum[v_{\mathcal{I}}(D) - u : D \in \Delta]$$

as required. Other cases are very similar. □

This is all very well. However, unfortunately, cut elimination fails for G∀Ł $+$ (cut) and cancellation elimination fails for G∀Ł $+$ (can), so we do not have an analytic calculus for this fragment of ∀Ł. For example, $(\exists x)(\forall y)(p(x) \to p(y))$ has the following proof in G∀Ł + (can):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(a) \Rightarrow p(a)}\;(id)}{(\forall z)p(z) \Rightarrow p(a)}\;(\forall\Rightarrow) \qquad \dfrac{\dfrac{}{p(a) \Rightarrow p(a)}\;(id) \qquad \dfrac{\dfrac{}{p(b) \Rightarrow p(b)}\;(id)}{(\forall z)p(z) \Rightarrow p(b)}\;(\forall\Rightarrow)}{(\forall z)p(z), p(a) \Rightarrow p(b), p(a)}\;(mix)}{(\forall z)p(z) \Rightarrow p(a) \to p(b), p(a)}\;(\Rightarrow\to)}{(\forall z)p(z) \Rightarrow (\forall y)(p(a) \to p(y)), p(a)}\;(\Rightarrow\forall)}{(\forall z)p(z) \Rightarrow (\exists x)(\forall y)(p(x) \to p(y)), p(a)}\;(\Rightarrow\exists)}{(\forall z)p(z) \Rightarrow (\exists x)(\forall y)(p(x) \to p(y)), (\forall z)p(z)}\;(\Rightarrow\forall)}{\Rightarrow (\exists x)(\forall y)(p(x) \to p(y))}\;(can)$$

But no G∀Ł-proof exists for this formula. A simple induction shows that there would be a branch in such a derivation where any hypersequent $\mathcal{G}$ on the branch satisfies the following property: $\mathcal{G}$ contains a sequent with an occurrence of a subformula of $(\exists x)(\forall y)(p(x) \to p(y))$ on the right that does not occur on the left in any sequent in $\mathcal{G}$. But then the branch cannot contain an initial sequent, a contradiction.

7 An Infinitary Calculus

To obtain a calculus that is complete for the full logic ∀Ł, we clearly need – as in the axiomatizations of Hay [15], Belluce and Chang [8, 7], and Hájek [13] – an infinitary rule. To establish the completeness of G∀Ł extended with such a rule, we make essential use of the approximate Herbrand Theorem and Skolemization results of earlier sections. The crucial step is to show that if a Herbrand disjunction of the prenex form of a formula C is ∀Ł-valid, then $\Rightarrow C$ is G∀Ł-derivable. We proceed in similar fashion to the proof of Lemma 2. That is, we show that by applying the rules of G∀Ł backwards to C, we arrive at propositional hypersequents equivalent to the appropriate Herbrand disjunctions. The complicating factors here are the presence of quantifiers deep within C and the use of rules to decompose the formula into sets of hypersequents.

Proposition 3. *Let* $(Q\bar{y})P(\bar{y})$ *be a prenex form of a formula* C *with Skolem form* $(\exists\bar{x})P^F(\bar{x})$*. If* $\models_{\text{Ł}} \bigvee_{i=1}^{n} P^F(\bar{t}_i)$*, then* $\vdash_{\mathsf{G}\forall\text{Ł}} \Rightarrow C$*.*

Proof. Let the sequence Γ_j of multisets of formulas with labelled occurrences of quantifiers be defined for $(Q\bar{y})P(\bar{y})$ exactly as in Lemma 2. Label the occurrences of $\forall$ and $\exists$ in C with the term $f(z_1, \ldots, z_n)$ labelling the corresponding shifted quantifier in

$(Q\bar{y})P(\bar{y})$ (noting that some occurrences of $\exists$ are transformed to $\forall$ while prenexing and vice versa). Recall that when substituting terms for variables in such labelled formulas, the substitution also applies to variables in the labels.

We define sets H_j of hypersequents as follows. Let $H_0 = \{\Rightarrow C\}$ and given H_j, let H_{j+1} be the result of applying the following operations to H_j exhaustively in order:

(i) Replace $\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta$ with $\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma, B \Rightarrow A, \Delta$.
(ii) Replace $\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta$ with $\mathcal{G} \mid \Gamma \Rightarrow \Delta$ and $\mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta$.
(iii) If $\mathcal{G} \mid \Gamma \Rightarrow (\forall x)B(x), \Delta$ is in the set and $f(\bar{t})$ labels $\forall$, add $\mathcal{G} \mid \Gamma \Rightarrow B(f(\bar{t})), \Delta$.
(iv) If $\mathcal{G} \mid \Gamma, (\forall x)B(x) \Rightarrow \Delta$ is in the set, add $\mathcal{G} \mid \Gamma, B(s) \Rightarrow \Delta$ for all $s \in U_j(P^F)$.
(v) If $\mathcal{G} \mid \Gamma, (\exists x)B(x) \Rightarrow \Delta$ is in the set and $f(\bar{t})$ labels $\exists$, add $\mathcal{G} \mid \Gamma, B(f(\bar{t})) \Rightarrow \Delta$.
(vi) If $\mathcal{G} \mid \Gamma \Rightarrow (\exists x)B(x), \Delta$ is in the set, add $\mathcal{G} \mid \Gamma \Rightarrow B(s), \Delta$ for all $s \in U_j(P^F)$.

Intuitively, what is happening here is that at each level j, we are applying backwards all the propositional rules of G∀Ł, then dealing with the quantifiers using terms from $U_j(P^F)$ in the same way as for Γ_j in Lemma 2. We need sets of hypersequents to cope with the fact that the $(\to\Rightarrow)$ rule has two premises.

Using the logical and structural rules of G∀Ł, for each j:

$$\text{if } \vdash_{\mathsf{G\forall Ł}} \mathcal{G} \text{ for all } \mathcal{G} \in H_{j+1}, \text{ then } \vdash_{\mathsf{G\forall Ł}} \mathcal{G} \text{ for all } \mathcal{G} \in H_j$$

Hence it is sufficient to show that $\vdash_{\mathsf{G\forall Ł}} \mathcal{G}$ for all $\mathcal{G} \in H_k$ for some $k \in \mathbb{N}$. In fact, it is enough to consider the "propositional part" of some H_k. Namely, for a hypersequent $\mathcal{G}$:

$$\mathit{prop}(\mathcal{G}) = [S \in \mathcal{G} : S \text{ contains only propositional formulas}]$$

It is sufficient to show that $\models_{Ł} \mathit{prop}(\mathcal{G})$ for all $\mathcal{G} \in H_k$ for some $k \in \mathbb{N}$. It then follows by the propositional completeness of G∀Ł and the external weakening rule (ew) that $\vdash_{\mathsf{G\forall Ł}} \mathcal{G}$ for all $\mathcal{G} \in H_k$.

Recall now from Lemma 2 that each Γ_j is a multiset of formulas containing only terms from $U_j(P^F)$. Let $\mathit{prop}(\Gamma_j) = [A \in \Gamma_j : A \text{ is a propositional formula}]$. Using the construction of the two sequences, for each $j \in \mathbb{N}$:

$$\models_{Ł} \bigvee \mathit{prop}(\Gamma_j) \quad \Longrightarrow \quad \models_{Ł} \mathit{prop}(\mathcal{G}) \text{ for all } \mathcal{G} \in H_j \tag{1}$$

But by Lemma 2, the left hand side holds for some $j \in \mathbb{N}$, so the result follows. Intuitively, (1) holds because the only difference between the interpretations of the sequences Γ_j and H_j is that in the latter, there is some rearranging of formulas using ∀Ł-equivalences. Formally this can be proved (rather tediously) by establishing a correspondence between formulas in Γ_j and subsets of H_j. ☐

Theorem 10. $\models_{Ł} A$ *iff* $\vdash_{\mathsf{G\forall Ł}} \Rightarrow A \oplus A^n$ *for all* $n \in \mathbb{N}^+$.

Proof. For the right-to-left direction, suppose that $\vdash_{\mathsf{G\forall Ł}} A \oplus A^n$ for all $n \in \mathbb{N}^+$. Then for all interpretations $\mathcal{I}$, $1 - 1/n \leq v_{\mathcal{I}}(A)$ for all $n \in \mathbb{N}^+$, so $v_{\mathcal{I}}(A) = 1$. For the left-to-right direction suppose that $\models_{Ł} A$. Let $(Q\bar{y})P(\bar{y})$ be a prenex form of A with Skolem form $(\exists\bar{x})P^F(\bar{x})$. By Theorem 3, for all $n \in \mathbb{N}^+$:

$$\models_{Ł}^{>1-1/n} \bigvee_{i=1}^{m} P^F(\bar{t}_i) \text{ for some } \bar{t}_1, \ldots, \bar{t}_m \in U(P^F)$$

For $j = 1 \ldots m$, let P^F_j be P^F with each Skolem function f replaced with distinguished new functions f_j. Then, by some arithmetical reasoning, for all $n \in \mathbb{N}^+$:

$$\models_{\text{Ł}} \bigvee_{i=1}^{m} (P^F_1 \oplus (P^F_2 \odot \ldots \odot P^F_n))(\bar{t}_i) \text{ for some } \bar{t}_1, \ldots, \bar{t}_m \in U(P^F)$$

Moreover, $P^F_1 \oplus (P^F_2 \odot \ldots \odot P^F_n)$ is $\forall$Ł-equivalent to the Skolem form of a prenex form of $A \oplus A^{n-1}$. Hence by the previous proposition $\vdash_{\mathsf{G}\forall\text{Ł}} \Rightarrow A \oplus A^{n-1}$. $\square$

We can use this result to obtain a proof system for $\forall$Ł with an infinitary rule (recalling that $[A]^n$ is the multiset consisting of n copies of A):

Corollary 3. *$\models_{\text{Ł}} A$ iff $\Rightarrow A$ is derivable in $\mathsf{G}\forall$Ł extended with the rule:*

$$\frac{\bot \Rightarrow [A]^n \ \textit{for all } n \in \mathbb{N}^+}{\Rightarrow A}$$

Example 3. Consider our earlier problematic formula $(\exists x)(\forall y)(p(x) \to p(y))$. Proving this in the above system requires an infinite number of derivations in $\mathsf{G}\forall$Ł. E.g in the case where $n = 2$, we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\bot \Rightarrow p(b) \to p(c)}\,(\bot\Rightarrow) \quad \dfrac{\dfrac{\dfrac{}{\bot \Rightarrow p(b)}\,(\bot\Rightarrow) \quad \dfrac{\dfrac{}{\bot \Rightarrow p(c)}\,(\bot\Rightarrow) \quad \dfrac{}{p(b) \Rightarrow p(b)}\,(id)}{\bot, p(b) \Rightarrow p(b), p(c)}\,(mix)}{\bot \Rightarrow p(b), p(b) \to p(c)}\,(\Rightarrow\to)}{\bot, p(a) \Rightarrow p(b), p(b) \to p(c)}\,(wl)}{\bot \Rightarrow p(a) \to p(b), p(b) \to p(c)}\,(\Rightarrow\to)}{\bot \Rightarrow p(a) \to p(b), (\forall y)(p(b) \to p(y))}\,(\Rightarrow\forall)}{\bot \Rightarrow p(a) \to p(b), (\exists x)(\forall y)(p(x) \to p(y))}\,(\Rightarrow\exists)}{\bot \Rightarrow (\forall y)(p(a) \to p(y)), (\exists x)(\forall y)(p(x) \to p(y))}\,(\Rightarrow\forall)}{\bot \Rightarrow (\exists x)(\forall y)(p(x) \to p(y)), (\exists x)(\forall y)(p(x) \to p(y))}\,(\Rightarrow\exists)$$

8 The One-Variable Fragment

Although infinitary rules are necessary to provide a calculus for the whole of $\forall$Ł, we may be able to do better in the case of particular fragments. Recall from Section 3 that the one-variable function-free fragment is decidable. We can obtain a cut-free hypersequent calculus for this fragment by liberalizing the eigenvariable condition for the rules $(\Rightarrow\forall)$ and $(\exists\Rightarrow)$ of $\mathsf{G}\forall$Ł. The idea is to allow for quantifier shifts that “could be performed lower down in the proof”.

Definition 3. *Let $\mathsf{G}\forall\text{Ł}_1$ be $\mathsf{G}\forall$Ł with the eigenvariable condition in Definition 2 changed to “a is either new or removed by $(\forall\Rightarrow)$ or $(\Rightarrow\exists)$ at a lower point in the proof”.*

The eigenvariable condition given here is global in the sense that it applies to whole proofs: the rules are not sound in isolation, but only as part of particular derivations.

Example 4. Consider the following proof of $(\exists x)(p(x) \to (\forall x)p(x))$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p(a) \Rightarrow p(a)}\ (id)}{p(a) \Rightarrow (\forall x)p(x)}\ (\Rightarrow \forall)}{\Rightarrow p(a) \to (\forall x)p(x)}\ (\Rightarrow \to)_1}{\Rightarrow (\exists x)(p(x) \to (\forall x)p(x))}\ (\Rightarrow \exists)$$

The introduction of a in the application of $(\Rightarrow \forall)$ is justified by the fact that a is removed by $(\Rightarrow \exists)$ two lines further down in the proof. In fact, the subproof ending with $p(a) \Rightarrow (\forall x)p(x)$ is not allowed in isolation: rightly so, since this sequent is not $\forall$Ł-valid.

Theorem 11. *Let A be a one-variable function-free formula. Then $\vdash_{\mathsf{G\forall \text{Ł}_1}} \Rightarrow A$ iff $\models_{\text{Ł}} A$.*

Proof. Recall the translations A^+ and A^- for one-variable function-free formulas A given in Section 3. The ith occurrence of a quantifier Q in A is annotated as Q^i and A^+ is obtained by removing positive occurrences of $\forall^i$ and negative occurrences of $\exists^i$ and replacing the bound variable with a new variable a_i not occurring in A. Recall also that we have $\models_{\text{Ł}} A$ iff $\models_{\text{Ł}} A^+$.

Suppose now for the left-to-right direction that $\vdash_{\mathsf{G\forall \text{Ł}_1}} \Rightarrow A$. We can assume that we have a $\mathsf{G\forall \text{Ł}_1}$-derivation where the variable introduced by a positive occurrence of $\forall^i$ or negative occurrence of $\exists^i$ is a_i. But then we obtain a $\mathsf{G\forall \text{Ł}}$-derivation of A^+ by removing these quantifier occurrences and applications of $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$ and replacing each occurrence of x bound by a removed quantifier $\forall^i$ or $\exists^i$ by a_i. By the soundness of $\mathsf{G\forall \text{Ł}}$, $\models_{\text{Ł}} A^+$. Hence also $\models_{\text{Ł}} A$.

For the right-to-left direction, suppose that $\models_{\text{Ł}} A$. Let the prenex form of A^+ (which contains only positive occurrences of $\exists^i$ and negative occurrences of $\forall^i$) be $(\exists \bar{x})P(\bar{x})$. Let us treat the free variables a_i now as constants. Then $\models_{\text{Ł}} (\exists \bar{x})P(\bar{x})$ and by the approximate Herbrand theorem:

$$\models_{\text{Ł}} P_1(\bar{t}_1) \vee \ldots \vee P_n(\bar{t}_n) \quad \text{for some } \bar{t}_1, \ldots, \bar{t}_n \in U(P)$$

where $U(P)$ contains only the a_i and perhaps constants occurring in A.

Hence, by the propositional completeness of $\mathsf{G\forall \text{Ł}}$:

$$\vdash_{\mathsf{G\forall \text{Ł}}} \Rightarrow P_1(\bar{t}_1) \mid \ldots \mid \Rightarrow P_n(\bar{t}_n)$$

To prove $\Rightarrow A$ we apply (ec) upwards n times to obtain $\Rightarrow A \mid \ldots \mid \Rightarrow A$. We then mimic the proof of $\Rightarrow P_1(\bar{t}_1) \mid \ldots \mid \Rightarrow P_n(\bar{t}_n)$ making sure that we choose the appropriate variables when we encounter occurrences of $\forall$ and $\exists$. For $(\forall \Rightarrow)$ and $(\Rightarrow \exists)$, this just entails choosing the appropriate a_i. For occurrences of $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$, we also need a variable. A new variable is fine. If the required variable is some a_i, then we know that it must be removed by an occurrence of $(\forall \Rightarrow)$ and $(\Rightarrow \exists)$ at some point in the derivation. Hence the relaxed eigenvariable condition is satisfied. $\square$

We remark finally that this approach works also to obtain Gentzen systems – decision procedures even – for other fragments such as the $\forall\exists$-function-free formulas of $\forall$Ł.

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