

# JSJ-Decompositions of Coxeter Groups over Virtually Abelian Splittings

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## Abstract

The idea of “JSJ-decompositions” for 3-manifolds began with work of Waldhausen and was developed later through work of Jaco, Shalen and Johansen. It was shown that there is a finite collection of 2-sided, incompressible tori that separate a closed irreducible 3-manifold into pieces with strong topological structure.

Sela introduced the idea of JSJ-decompositions for groups, an idea that has flourished in a variety of directions. The general idea is to consider a certain class  $\mathcal{G}$  of groups and splittings of groups in  $\mathcal{G}$  by groups in another class  $\mathcal{C}$ . E.g. Rips and Sela considered splittings of finitely presented groups by infinite cyclic groups. For an arbitrary group  $G$  in  $\mathcal{G}$ , the goal is produce a unique reduced graph of groups decomposition  $\Psi$  of  $G$  with edge groups in  $\mathcal{C}$  so that  $\Psi$  reveals all reduced graph of groups decompositions of  $G$  with edge groups in  $\mathcal{C}$ . More specifically, if  $H$  is a vertex group of  $\Psi$  then either there is no  $\mathcal{C}$ -group that splits both  $G$  and  $H$ , or  $H$  has a special “surface group-like” structure. Vertex groups of the second type are standardly called orbifold groups.

For a finitely generated Coxeter system  $(W, S)$ , we produce a JSJ-decomposition  $\Psi$  for splittings of  $W$  over virtually abelian subgroups. We show  $\Psi$  is unique and each vertex and edge group is generated by a subset of  $S$  (and so  $\Psi$  is “visual”). The construction of  $\Psi$  is algorithmic. If  $V \subset S$  generates an orbifold (vertex) group then  $V = T \cup M$ , where  $\langle M \rangle$  is virtually abelian,  $\langle T \rangle$  is virtually a closed surface group or virtually free and  $\langle V \rangle = \langle M \rangle \times \langle T \rangle$ .

# 1 Introduction

The theory of JSJ-decompositions has its origins in the work of Waldhausen [16] on characteristic submanifolds of a 3-manifold and later work of Jaco-Shalen [8] and Johansen [9]. For a closed, irreducible, oriented 3-manifold there is a finite collection of embedded 2-sided incompressible tori that separate the manifold into pieces, each of which is a Seifert fibered space or an atoroidal and acylindrical space. This gives a graph of groups decomposition of the fundamental group with edge groups free abelian of rank 2.

Sela [14] introduced the notion of JSJ-decomposition for a general class of groups and showed that word hyperbolic groups have JSJ-decompositions over infinite cyclic splittings. Rips-Sela [13] generalize this to finitely presented groups. Dunwoody-Sageev [6] and then Fujiwara-Papasoglu [7] gave JSJ-decompositions for finitely presented groups over slender splittings.

A group is *slender* if all of its subgroups are finitely generated. The class of slender groups is contained in the class of *small* groups which are defined in terms of actions on trees. If a group contains a non-abelian free group it is not small. Coxeter groups containing no non-abelian free group are in fact virtually abelian and decompose in a special way amenable to our results (see theorem 13).

In analogy with the 1-ended assumptions of Rips-Sela [13], and following Dunwoody-Sageev [6] directly, we define the class of minimal virtually abelian splitting subgroups of  $W$ . If a Coxeter group splits over a minimal subgroup that contains no non-abelian free group, then the splitting subgroup is in fact virtually abelian. Hence for our purposes, there is no difference between (minimal) splittings over small, slender or virtually abelian groups and we only consider splittings of Coxeter groups over virtually abelian subgroups.

If  $(W, S)$  is a finitely generated Coxeter system, a graph of groups decomposition  $\Psi$  of  $W$  is *visual* if each edge and vertex group is generated by some subset of  $S$  and the bonding maps are inclusions. The main theorem of [11] states that for any graph of groups decomposition  $\Lambda$  of  $W$  there is a visual decomposition  $\Psi$  such that each vertex (respectively edge) group of  $\Psi$  is conjugate to a subgroup of a vertex (respectively edge) group of  $\Lambda$ . This result is used extensively in this paper and the basics of visual decompositions are reviewed in section 2.

Our construction of a JSJ-decomposition of a finitely generated Coxeter group  $W$ , with virtually abelian edge groups begins with a graph of groups decomposition  $\Psi_1$  of  $W$  with edge groups that are minimal virtually abelian

splitting subgroups of  $W$  and such that  $\Psi_1$  is a maximal such decomposition without edge groups that are “crossing splitters”. We also show that  $\Psi_1$  is unique (up to conjugate vertex groups) and visual. We call  $\Psi_1$  a *level 1 JSJ-decomposition of  $W$  with virtually abelian edge groups*. If  $V \subset S$  and  $\langle V \rangle$  is a vertex group of  $\Psi_1$ , then  $\langle V \rangle$  may contain virtually abelian subgroups that split  $\langle V \rangle$  and  $W$  non-trivially. Minimal splitting subgroups of this type are not necessarily minimal virtually abelian splitting subgroups of  $W$ . It is shown that any graph of groups decomposition of  $\langle V \rangle$  with such edge groups is compatible with  $\Psi_1$  and a maximal such decomposition that avoids crossing splitters, is unique (up to conjugate vertex groups) and visual. Replacing all such vertex groups of  $\Psi_1$  by such graph of groups decompositions gives  $\Psi_2$ , a *level 2 JSJ-decomposition of  $W$  with virtually abelian edge groups*. Continuing, we eventually have a visual graph of groups decomposition  $\Psi$  such that if  $\langle V \rangle$  is a vertex group of  $\Psi$ , then the only virtually abelian splitting subgroups of  $\langle V \rangle$  (that also split  $W$ ) are crossing. We call  $\Psi$  a *JSJ-decomposition of  $W$  with respect to virtually abelian splittings* and show that  $\Psi$  is unique (up to conjugate vertex groups). If  $\Lambda$  and  $\Phi$  are graph of groups decompositions of a group  $W$ , then say *the decomposition of  $\Lambda$  induced by  $\Phi$  is compatible with  $\Lambda$*  if for each vertex group  $V$  of  $\Lambda$ , the decomposition of  $V$  induced by the action of  $V$  on the Bass-Serre tree for  $\Phi$  is compatible with  $\Lambda$ . In sections 7 and 8, we prove results that imply our main theorem:

**Theorem 1 (Main Theorem)** *Suppose  $(W, S)$  is a finitely generated Coxeter system and  $\Psi$  the JSJ-decomposition of  $W$  with virtually abelian edge groups. Then:*

1.  $\Psi$  is visual, unique and algorithmically defined.
2. If  $\Phi$  is a graph of groups decomposition of  $W$  with virtually abelian edge groups then the decomposition of  $\Psi$  induced by  $\Phi$  is compatible with  $\Psi$ .
3. If both  $W$  and a vertex group  $\langle V \rangle$  of  $\Psi$  ( $V \subset S$ ) split nontrivially over a virtually abelian subgroup of  $\langle V \rangle$ , then  $\langle V \rangle$  decomposes as  $\langle T \rangle \times \langle M \rangle$  where  $T \cup M = V$ ,  $M$  generates a virtually abelian group and the presentation diagram of  $T$  is either a loop of length  $\geq 4$  (in which case  $T$  generates a group that is virtually a closed surface group) or the presentation diagram of  $T$  is a disjoint union of vertices and simple paths (in which case  $T$  generates a virtually free group with graph of groups decomposition such that each vertex group is either  $\mathbb{Z}_2$  or finite dihedral and each edge group is either trivial or  $\mathbb{Z}_2$ ).

Vertex groups of the type described by part 3 of theorem 1 are called *orbifold* vertex groups. If  $H$  is a non-orbifold vertex group of our JSJ-decomposition  $\Psi$ , then  $H$  does not split non-trivially over a virtually abelian subgroup that also splits  $W$  non-trivially. In particular, if  $\Phi$  is another graph of groups decomposition of  $W$  with virtually abelian edge groups, then  $H$  is a subgroup of a conjugate of a vertex group of  $\Phi$ .

The construction of our JSJ-decompositions with virtually abelian edge groups is algorithmic. Given the presentation diagram  $\Gamma$  of a Coxeter system  $(W, S)$ , theorem 13 allows us to determine the subsets of  $S$  that generate virtually abelian subgroups. Those that separate  $\Gamma$ , algebraically split  $W$ . A result in section 4 allows us to easily decide which of these are the visual minimal virtually abelian splitting subgroups of  $(W, S)$  at all stages of the construction of the  $i^{\text{th}}$ -level JSJ-decompositions. It is equally easy to “visually” determine which of these splitting subgroups are crossing and hence build JSJ-decompositions. Our main result distinguishes the two types of vertex groups of our JSJ-decompositions. Those with no crossing subgroups are indecomposable with respect to virtually abelian splittings of  $W$  and those with crossing subgroups which are traditionally called orbifold vertex groups.

## 2 Basic Facts and Background Results

A thorough discussion of graphs of groups decompositions of Coxeter groups is given in [11]. We briefly discuss the aspects of this theory necessary to this paper. Every Coxeter group has a set of order 2 generators and so there is no non-trivial map of a Coxeter group to  $\mathbb{Z}$ . In particular, no Coxeter group is an HNN extension of any sort and any graph of groups decomposition of a Coxeter group has graph a tree. Hence the decompositions of Coxeter groups are a straightforward generalization of amalgamated product decompositions. For a graph of groups decomposition  $\Lambda$  of a group  $G$ , the Bass-Serre tree  $T$  for  $\Lambda$  has vertices (respectively edges) the cosets  $wV$  where  $w \in G$  and  $V$  is a vertex (respectively edge) group of  $\Lambda$ . There is an left action of  $G$  on  $T$  and an element  $g$  of  $G$  stabilizes the coset  $wV$  iff  $g \in wVw^{-1}$ . If  $V$  is a vertex of  $\Lambda$  with vertex group  $\Lambda(V)$ , and  $\Phi$  a graph of groups decomposition of  $\Lambda(V)$ , then  $\Phi$  is *compatible* with  $\Lambda$  if for each edge  $E$  of  $\Lambda$  adjacent to  $V$ ,  $\Lambda(E)$  is contained in a  $\Lambda(V)$ -conjugate of a vertex group of  $\Phi$ . In this case  $V$  can be replaced by  $\Phi$  to produce a finer graph of groups decomposition of

$G$ . A graph of groups decomposition  $\Lambda$  is *reduced* if no edge between distinct vertices has edge group the same as an end point vertex group. If a graph of groups is not reduced, we may collapse a vertex group across an edge, where the edge group is the same as the endpoint vertex group, giving a smaller graph of groups decomposition of the original group.

**Lemma 2** *Suppose  $\Lambda$  is a reduced graph of groups decomposition of a group  $G$ , the underlying graph for  $\Lambda$  is a tree,  $U$  and  $V$  are vertices of  $\Lambda$ , and  $g\Lambda(U)g^{-1} = \Lambda(V)$  for some  $g \in G$  then  $U = V$  and  $g \in \Lambda(U)$ .  $\square$*

**Lemma 3** *Suppose  $\Psi$  is a graph of groups decomposition of a group  $G$ ,  $V$  is a finitely generated vertex group of  $\Psi$ , and  $\mathcal{E}$  is a collection of subgroups of  $V$  such that for any  $K \in \mathcal{E}$ ,  $V$  splits non-trivially and  $\Psi$ -compatibly over  $K$ . If  $\Lambda$  is a graph of groups decomposition of  $V$  with edge groups in  $\mathcal{E}$ , then  $\Lambda$  is compatible with  $\Psi$ .  $\square$*

The following result easily follows from the combinatorics of group actions on trees or more practically from the exactness of the Mayer-Vietoris sequence for a pair of groups.

**Lemma 4** *Suppose a group  $G$  splits as  $A *_C B$ . If there is no non-trivial homomorphism from  $G$  or  $C$  to  $\mathbb{Z}$ , then there is no non-trivial homomorphism from  $A$  or  $B$  to  $\mathbb{Z}$ . In particular, if  $\Lambda$  is a graph of groups decomposition of a Coxeter group and no edge group of  $\Lambda$  maps non-trivially to  $\mathbb{Z}$ , then no vertex group of  $\Lambda$  maps non-trivially to  $\mathbb{Z}$ .  $\square$*

We take a *Coxeter presentation* to be given as

$$P = \langle S : (st)^{m(s,t)} \ (s, t \in S, m(s, t) < \infty) \rangle$$

where  $m : S^2 \rightarrow \{1, 2, \dots, \infty\}$  is such that  $m(s, t) = 1$  iff  $s = t$ , and  $m(s, t) = m(t, s)$ . In the group with this presentation, the elements of  $S$  represent distinct elements of order 2 and a product  $st$  of generators has order  $m(s, t)$ . A *Coxeter group*  $W$  is a group having a Coxeter presentation and a *Coxeter system*  $(W, S)$  is a Coxeter group  $W$  with generating subset  $S$  corresponding to the generators in a Coxeter presentation of  $W$ . When the order of the product of a pair of generators is infinite there will be no defining relator for that pair of generators and we will say that the generators are *unrelated*. Our basic reference for Coxeter groups is Bourbaki [2]. A *special* or *visual*

subgroup for a Coxeter system  $(W, S)$ , is a subgroup of  $W$  generated by a subset of  $S$ . If  $W'$  is the visual subgroup generated by  $S' \subseteq S$  in a Coxeter system  $(W, S)$ , then  $(W', S')$  is also a Coxeter system. More specifically the following result (see [2]) is fundamental to the study of Coxeter groups.

**Proposition 5** *Suppose  $(W, S)$  is a Coxeter system and  $P = \langle S : (st)^{m(s,t)}$  for  $m(s, t) < \infty \rangle$  (where  $m : S^2 \rightarrow \{1, 2, \dots, \infty\}$ ) is a Coxeter presentation for  $W$ . If  $A \subset S$ , then  $(\langle A \rangle, A)$  is a Coxeter system with Coxeter presentation  $\langle A : (st)^{m'(s,t)}$  for  $m'(s, t) < \infty \rangle$  (where  $m' = m|_{A^2}$ ).  $\square$*

Given a group  $G$  and a generating set  $S$ , an  $S$ -geodesic for  $g \in G$  is a shortest word in  $S \cup S^{-1}$  such that the product of the letters of this word is  $g$ . The number of letters in an  $S$ -geodesic for  $g$  is the  $S$ -length of  $g$ . An important combinatorial fact about geodesics for a Coxeter system is called the “deletion condition”.

**Proposition 6 (The Deletion Condition)** *Suppose  $(W, S)$  is a Coxeter system and  $w = a_1 \cdots a_n$  for  $a_i \in S$ . If  $a_1 \cdots a_n$  is not geodesic then there are indices  $i < j$  in  $\{1, 2, \dots, n\}$  such that  $w = a_1 \cdots a_{i-1} a_{i+1} \cdots a_{j-1} a_{j+1} \cdots a_n$ . I.e.  $a_i$  and  $a_j$  can be deleted.  $\square$*

The information given by a Coxeter presentation may be conveniently expressed in the form of a labeled graph. We define the *presentation diagram* of the system  $(W, S)$  to be the labeled graph  $\Gamma(W, S)$  with vertex set  $S$ , and an (undirected) edge labeled  $m(s, t)$  between distinct vertices  $s$  and  $t$  when  $m(s, t) < \infty$ . The connected components of the presentation diagram  $\Gamma(W, S)$  correspond to visual subgroups which are the factors in a free product decomposition of  $W$ . In contrast, a *Coxeter graph* has vertex set  $S$  and labeled edges when  $m(s, t) \neq 2$ . The components of a Coxeter graph corresponding to direct product factors of  $W$ . By proposition 5, a presentation diagram of a visual subgroup of  $W$  generated by a subset  $S' \subseteq S$  is the induced subgraph of  $\Gamma(W, S)$  with vertex set  $S'$ .

Suppose  $\Gamma(W, S) = \Gamma_1 \cup \Gamma_2$  is a union of induced subgraphs and let  $\Gamma_0 = \Gamma_1 \cap \Gamma_2$  (so vertices and edges of  $\Gamma(W, S)$  are in  $\Gamma_1$  or  $\Gamma_2$  or both, and  $\Gamma_0$  is the induced subgraph consisting of the vertices and edges in both). Equivalently, suppose  $\Gamma_0$  is an induced subgraph with  $\Gamma(W, S) - \Gamma_0$  having at least two components,  $\Gamma_1$  is  $\Gamma_0$  together with some of these components and  $\Gamma_2$  is  $\Gamma_0$  together with the other components. We say in this case that  $\Gamma_0$  *separates*

$\Gamma(W, S)$  (separates it into at least two components). Then it is evident from the Coxeter presentation that  $W$  is an amalgamated product of visual subgroups corresponding to  $\Gamma_1$  and  $\Gamma_2$  over the visual subgroup corresponding to  $\Gamma_0$ . Amalgamated product decompositions with visual factors and visual amalgamated subgroup are easily seen in the presentation diagram and we call such an amalgamated product a *visual splitting* of  $W$ .

We say that  $\Psi$  is a *visual graph of groups decomposition* of  $W$  (for a given Coxeter system  $(W, S)$ ), if each vertex and edge group of  $\Psi$  is visual for  $(W, S)$ , the injections of each edge group into its endpoint vertex groups are given simply by inclusion, and the fundamental group of  $\Psi$  is isomorphic to  $W$  by the homomorphism induced by the inclusion map of vertex groups into  $W$ . A sequence of compatible visual splittings of  $W$  will result in such a decomposition. In [11], we study general graph of groups decompositions of Coxeter groups and how these are related to visual graph of groups decompositions. The main result of [11] shows that an arbitrary graph of groups decomposition of a Coxeter group can be refined (in a certain sense) to a visual graph of groups decomposition.

**Theorem 7** *Suppose  $(W, S)$  is a Coxeter system and  $W$  is a subgroup of the fundamental group of a graph of groups  $\Lambda$ . Then  $W$  has a visual graph of groups decomposition  $\Psi$  where each vertex group of  $\Psi$  is a subgroup of a conjugate of a vertex group of  $\Lambda$ , and each edge group of  $\Psi$  is a subgroup of a conjugate of an edge group of  $\Lambda$ . Moreover,  $\Psi$  can be taken so that each visual subgroup of  $W$  that is a subgroup of a conjugate of a vertex group of  $\Lambda$  is a subgroup of a vertex group of  $\Psi$ .  $\square$*

The following three results are important technical facts proved in [11].

**Lemma 8** *Suppose  $(W, S)$  is a Coxeter system. A graph of groups  $\Psi$  with graph a tree, where each vertex group and edge group is a visual subgroup for  $(W, S)$  and each edge map is given by inclusion, is a visual graph of groups decomposition of  $W$  iff each edge in the presentation diagram of  $W$  is an edge in the presentation diagram of a vertex group and, for each generator  $s \in S$ , the set of vertices and edges with groups containing  $s$  is a nonempty subtree in  $\Psi$ .  $\square$*

If  $\Psi$  is a visual graph of groups decomposition for the Coxeter system  $(W, S)$ , it is convenient to label the vertices of  $\Psi$  by the subsets of  $S$  that

generate the corresponding vertex groups. So if  $Q \subset S$  is a vertex of  $\Psi$ , then  $\Psi(Q) = \langle Q \rangle$ . The same convention is not used for edge groups as distinct edges may have the same edge group.

Separation properties of various subsets of a presentation diagram  $\Gamma(W, S)$  for a Coxeter system  $(W, S)$  are frequently analyzed in this paper. If  $A$  and  $B$  are subsets of  $\Gamma$  we say  $A$  separates  $B$  in  $\Gamma$  if there are points  $b_1$  and  $b_2$  of  $B - A$  such that any path in  $\Gamma$  from  $b_1$  to  $b_2$  intersects  $A$ .

**Lemma 9** *Suppose  $(W, S)$  is a Coxeter system,  $S$  is finite,  $\Psi$  is a visual graph of groups decomposition of  $W$ ,  $E'$  is an edge of  $\Psi$  and  $\Psi(E') = \langle E \rangle$  for  $E \subset S$ . If  $\{x, y\} \subset S - E$ , and  $x \in X$  and  $y \in Y$  for  $X$  and  $Y$  vertices of  $\Psi$  on opposite sides of  $E'$ , then  $E$  separates  $x$  and  $y$  in  $\Gamma(W, S)$ .  $\square$*

**Lemma 10** *Suppose  $(W, S)$  is a finitely generated Coxeter system, and  $\Psi$  is a visual graph of groups decomposition of  $W$ . If  $C$  is a complete subset of the presentation diagram  $\Gamma(W, S)$ , then there is a vertex  $V$  of  $\Psi$  such that  $C \subset V$ .  $\square$*

In this paper we require more than just the statement of theorem 7. The technique to produce a visual decomposition is easy to describe and useful to the constructions in this paper. Under the hypothesis of theorem 7 let  $T$  be the Bass-Serre tree for  $\Lambda$ . In the proof of theorem 7, it is shown that  $W$  has a visual graph of groups decomposition with graph  $T$  and vertex (respectively edge) group at  $gV$  generated by the subset of  $S$  that stabilizes  $gV$ . Since  $S$  is finite, this visual graph of groups reduces to a finite reduced graph of groups decomposition of  $W$  satisfying the conclusion of theorem 7. In this paper we make repeated use of this construction and refer to it as *the visual graph of groups given by the construction for theorem 7*.

The next lemma follows from a result of Kilmoyer (see section 4 of [11]).

**Lemma 11** *Suppose  $(W, S)$  is a Coxeter system,  $I, J \subset S$ , and  $d$  is a minimal length double coset representative in  $\langle I \rangle w \langle J \rangle$ . Then  $\langle I \rangle \cap d \langle J \rangle d^{-1} = \langle K \rangle$  for  $K = I \cap (dJd^{-1})$  and,  $d^{-1} \langle K \rangle d = \langle J \rangle \cap (d^{-1} \langle I \rangle d) = \langle K' \rangle$  for  $K' = J \cap d^{-1} I d = d^{-1} K d$ . In particular, if  $w = idj$  for  $i \in \langle I \rangle$  and  $j \in \langle J \rangle$  then  $\langle I \rangle \cap w \langle J \rangle w^{-1} = i \langle K \rangle i^{-1}$  and  $\langle J \rangle \cap w^{-1} \langle I \rangle w = j^{-1} \langle K' \rangle j$ .  $\square$*

### 3 Decomposing Coxeter Groups With No Non-Abelian Free Subgroup

The Euclidean simplex reflection groups and irreducible finite Coxeter groups are catalogued in Coxeter's book [3]. Euclidean simplex groups are infinite and virtually abelian. In his thesis [10], D. Krammer classifies free abelian subgroups of Coxeter groups. If a Coxeter group contains no non-abelian free group, the Coxeter group is virtually free abelian and decomposes as a direct product of finite Coxeter groups and Euclidean simplex groups in a special way - a result well-known to experts. We include an elementary proof of this result, using the following lemma (which is a direct consequence of theorems 7.2.2, 7.3.1, 12.1.19 and exercise 12.1.14 of J. Ratcliffe's book [12]).

**Lemma 12** *Consider the collection of Coxeter systems  $(W, S)$  such that*

1.  $\Gamma(W, S)$  is complete,
2.  $W$  does not decompose as  $\langle A \rangle \times \langle B \rangle$  for  $A$  and  $B$  non-trivial subsets of  $S$ , and
3. for each  $s \in S$ ,  $\langle S - \{s\} \rangle$  is either finite or Euclidean.

*Then  $W$  is finite, Euclidean or contains a free subgroup of rank 2.  $\square$*

**Theorem 13** *Suppose  $(W, S)$  is a finitely generated Coxeter system and  $W$  does not contain a non-abelian free group. Then  $S$  is the disjoint union of sets which commute with one another and each generates a finite group or an Euclidean simplex group.*

**Proof:** If  $a, b \in S$  such that  $m(a, b) = \infty$ , let  $C = S - \{a, b\}$ . Then  $W = \langle \{a\} \cup C \rangle *_{\langle C \rangle} \langle \{b\} \cup C \rangle$ . The index of  $\langle C \rangle$  in both  $\langle \{a\} \cup C \rangle$  and  $\langle \{b\} \cup C \rangle$  is 2, since,  $W$  contains no free group of rank 2. Hence  $\langle C \rangle$  is normal in  $\langle \{a\} \cup C \rangle$ . For each  $c \in C$ ,  $aca \in \langle C \rangle$ . Any geodesic in  $\langle C \rangle$  uses only letters in  $C$ , and by the deletion condition,  $aca = c$ . Thus  $a$  commutes with  $C$  as does  $b$ . The group  $\langle a, b \rangle$  splits off as a direct factor of  $W$ . Hence we may assume  $\Gamma(W, S)$  is complete (and infinite). Note that condition 1) of the lemma is satisfied.

Suppose  $(W, S)$  is a counterexample to the theorem, with  $|S|$  small as possible. Note that  $|S| \geq 3$ , and  $W$  does not visually decompose as a non-trivial direct product, so that condition 2) of the lemma is satisfied. If

$s \in S$ , we have  $\langle S - \{s\} \rangle$  is a visual product of a finite group and Euclidean simplex groups. Choose  $a$  and  $b$  distinct elements of  $S$ . Write  $\langle S - \{a\} \rangle = \langle F_a \rangle \times \langle E_1 \rangle \times \cdots \times \langle E_p \rangle$  (so  $F_a \cup (\cup_{i=1}^p E_i) = S - \{a\}$ ) where  $\langle F_a \rangle$  is finite and each  $\langle E_i \rangle$  is Euclidean. Similarly write  $\langle S - \{b\} \rangle = \langle F_b \rangle \times \langle K_1 \rangle \times \cdots \times \langle K_q \rangle$ . If  $b \in E_i$  for some  $i$ , then assume  $i = 1$ . If  $p > 1$ , then  $E_2 \subset K_j$  for some  $j$ . For any proper subset  $K$  of  $K_j$ ,  $\langle K \rangle$  is finite, and so  $E_2 = K_j$ . But then  $E_2$  commutes with  $S - E_2$  which is impossible. Hence  $\{p, q\} \subset \{0, 1\}$ .

If  $F_a \neq \emptyset$  and  $E_1 \neq \emptyset$ , we may choose  $b \in F_a$ , so that  $\{a, b\} \cap E_1 = \emptyset$ . This implies  $E_1 = K_1$  and  $E_1$  commutes with  $S - E_1 (= F_a \cup \{a\})$ , which is impossible. Hence either  $F_a$  or  $E_1$  is empty. Since  $a$  is arbitrary in  $S$ , condition 3) of the previous lemma is satisfied.  $\square$

## 4 Minimal Virtually Abelian Splitting Subgroups

If  $\Lambda$  is a graph of groups decomposition of a group  $W$  and  $G$  is a vertex group of  $\Lambda$ , then a virtually abelian subgroup  $A$  of  $G$  is a *minimal virtually abelian splitting subgroup* for  $(\Lambda, G)$  if  $G$  splits non-trivially and compatibly with  $\Lambda$  over  $A$ , and there is no virtually abelian subgroup  $B$  of  $W$  such that  $G$  splits non-trivially and compatibly with  $\Lambda$  over  $B$ , and  $B \cap A$  has infinite index in  $A$  and finite index in  $B$ .

For a Coxeter system  $(W, S)$ ,  $\Psi$  a visual graph of groups decomposition for  $(W, S)$  and  $G$  a vertex group of  $\Psi$ , let  $\mathcal{C}(\Psi, G)$  be the set of virtually abelian subgroups of  $G$  that split  $G$  non-trivially and  $\Psi$ -compatibly. Let  $M(\Psi, G)$  be the set of minimal virtually abelian splitting subgroups for  $(\Psi, G)$ .

If  $\Psi$  is the trivial graph of groups decomposition for  $(W, S)$  (with one vertex), then define  $\mathcal{C}(W, S) \equiv \mathcal{C}(\Psi, W)$  and  $M(W, S) \equiv M(\Psi, W)$ . Observe:

1. If a vertex group  $G$  of  $\Psi$  has more than 1-end, then each member of  $M(\Psi, G)$  is a finite group.
2. For a given finitely generated Coxeter group  $W$ , the ranks of the virtually abelian subgroups of  $W$  are bounded.
3. If  $A \subset S$  and  $\langle A \rangle \in M(\Psi, G)$  then  $A$  satisfies the conclusion of theorem 13. Hence  $A$  is the disjoint union of sets that commute with one another such that (at most) one generates a finite group and each other generates an Euclidean simplex group.

If  $(W, S)$  is a Coxeter system and  $A \subset S$  is such that  $\langle A \rangle$  is virtually abelian, define  $E(A)$  to be the set of generators of the Euclidean factors of the visual direct product decomposition of  $\langle A \rangle$  given by theorem 13.

If  $A$  is a minimal virtually abelian splitting subgroup of a Coxeter group, then by theorem 7,  $A$  contains a subgroup of finite index which is isomorphic to a Coxeter group. Hence there is no non-trivial homomorphism of  $A$  to  $\mathbb{Z}$ . Lemma 4 implies the next result.

**Lemma 14** *Suppose  $W$  is a finitely generated Coxeter group and  $\Lambda$  is a graph of groups decomposition of  $W$  with minimal virtually abelian splitting subgroups as edge groups. Then  $\Lambda$  is a tree and no vertex group of  $\Lambda$  maps non-trivially to  $\mathbb{Z}$ .  $\square$*

If  $\Psi$  is a visual graph of groups decomposition for the Coxeter system  $(W, S)$ ,  $V(\subset S)$  is a vertex of  $\Psi$  and  $\Psi_1$  is a visual decomposition for  $(\langle V \rangle, V)$ , then  $\Psi_1$  is *visually compatible* with  $\Psi$  if for each edge group  $\langle E \rangle$  ( $E \subset S$ ) of an edge of  $\Psi$  adjacent to  $V$ ,  $E$  is a subset of  $K$  for some vertex  $K(\subset V)$  of  $\Psi_1$ . In particular, if  $A \subset V$  is such that  $\langle A \rangle$  is an edge group of  $\Psi_1$ , then we say  $\langle A \rangle$  *splits  $V$  visually and compatibly with  $\Psi$* .

**Lemma 15** *Suppose  $(W, S)$  is a finitely generated Coxeter system,  $\Psi$  is a reduced visual graph of groups decomposition for  $(W, S)$ ,  $V \subset S$  is a vertex of  $\Psi$ , and  $\Lambda$  is a reduced graph of groups decomposition of  $\langle V \rangle$  such that each edge group of  $\Lambda$  is in  $M(\Psi, \langle V \rangle)$ . Let  $\Psi'$  be the reduced visual decomposition for  $\Lambda$  given by the construction for theorem 7, then  $\Psi'$  is visually compatible with  $\Psi$ , each edge group of  $\Psi'$  is in  $M(\Psi, \langle V \rangle)$  and if  $U \subset V$  such that  $\langle U \rangle$  is an edge group of  $\Psi'$ , then  $U$  separates  $V$  in  $\Gamma(W, S)$ .*

**Proof:** By lemma 3,  $\Lambda$  is compatible with  $\Psi$ . So if  $E'$  is an edge of  $\Psi$  adjacent to  $V$  and  $E \subset S$  is such that  $\langle E \rangle = \Psi(E')$ , then  $\langle E \rangle$  is a subgroup of a  $\langle V \rangle$ -conjugate of a vertex group of  $\Lambda$ . Hence  $E$  stabilizes a vertex of the Bass-Serre tree for  $\Lambda$ . But then by the construction for theorem 7,  $E$  is a subset of a vertex group of  $\Psi'$  and  $\Psi'$  is visually compatible with  $\Psi$ . Each edge group of  $\Psi'$  is conjugate to a subgroup of an edge group of  $\Lambda$  and so must be in  $M(\Psi, \langle V \rangle)$ . Suppose  $A, B \subset V$  are distinct vertices of  $\Psi'$  adjacent to the edge  $U$  of  $\Psi'$ . There exists  $a \in A - U$  and  $b \in B - U$ . If  $\Psi''$  is the visual graph of groups decomposition obtained from  $\Psi$  by replacing the vertex  $V$  by  $\Psi'$ , then applying lemma 9 to  $\Psi''$  shows  $U$  separates  $a$  and  $b$  in  $\Gamma$ .  $\square$

The next lemma is a direct consequence of V. Deodhar's results in [4].

**Lemma 16** Suppose  $(W, S)$  is a Coxeter system  $A \subset S$ ,  $\langle A \rangle$  is infinite and there is no non-trivial  $F \subset A$  such that  $\langle F \rangle$  is finite and  $\langle A \rangle = \langle A - F \rangle \times \langle F \rangle$ . If  $w \in W$  such that  $w\langle A \rangle w^{-1} \subset \langle B \rangle$  for  $B \subset S$ , then  $uAu^{-1} = A \subset B$  for  $u$  the minimal length double coset representative of  $\langle B \rangle w \langle A \rangle$ . In particular, if  $w\langle A \rangle w^{-1} = \langle B \rangle$ , then  $A = B$ .  $\square$

Observe that if  $(W, S)$  is a Coxeter system, and  $W = \langle F \rangle \times \langle G \rangle = \langle H \rangle \times \langle I \rangle$  for  $F \cup G = S = H \cup I$ . Then  $W = \langle F \rangle \times \langle H - F \rangle \times \langle G \cap I \rangle$ .

**Proposition 17** Suppose  $(W, S)$  is a Coxeter system,  $A \subset S$  and  $\langle A \rangle = \langle B \rangle \times \langle C \rangle$  where  $B \cup C = A$ , and  $C$  is the (unique) largest such subset of  $A$  such that  $\langle C \rangle$  is finite. If  $w\langle A \rangle w^{-1} \subset \langle H \rangle$  for  $w \in W$  and  $H \subset S$ , then  $B \subset H$ . In particular, if  $\langle A \rangle$  and  $\langle H \rangle$  are virtually abelian then  $E(A) \subset E(H)$ .

**Proof:** Let  $d$  be a minimal length double coset representative of  $\langle H \rangle w \langle A \rangle$ . So  $d\langle A \rangle d^{-1} \subset \langle H \rangle$  and  $d\langle A \rangle d^{-1} = d\langle A \rangle d^{-1} \cap \langle H \rangle = \langle K \rangle$  for  $K = dAd^{-1} \cap H$ . Hence  $K = dAd^{-1}$  ( $d^{-1}Kd \subset A$  and generates  $\langle A \rangle$ ). Apply lemma 16.  $\square$

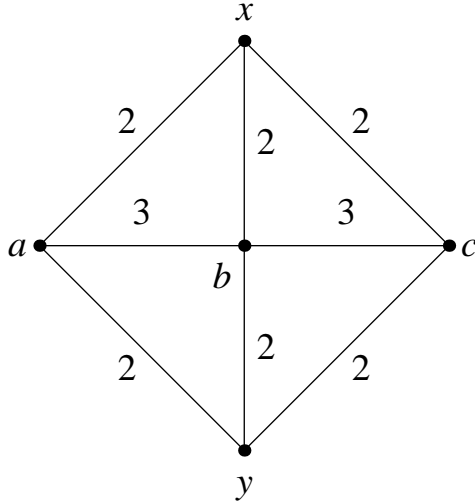


Figure 1:  $\Gamma(W, S)$

**Example 1.** Consider the Coxeter system  $(W, S)$  with presentation diagram shown in figure 1. Clearly,  $\langle x, y, b \rangle \in M(W, S)$  and the element  $bc$  conjugates

$\{x, y, b\}$  to  $\{x, y, c\}$ . So,  $\langle x, y, c \rangle \in M(W, S)$ . Hence a visual subgroup in  $M(W, S)$  need not separate the presentation diagram  $\Gamma(W, S)$ .

**Example 2.** The group with presentation diagram shown in figure 2, splits non-trivially and visually over both  $\langle a, b, c, d \rangle \in \mathcal{C}(W, S)$  and  $\langle a, c, e \rangle$ . As  $W$  is 1-ended and  $\langle a, c, e \rangle$  is 2-ended,  $\langle a, c, e \rangle \in M(W, S)$ . Hence  $\langle a, b, c, d \rangle \notin M(W, S)$ . Furthermore, no  $A \subset \{a, b, c, d\}$  is such that  $\langle A \rangle \in M(W, S)$ .

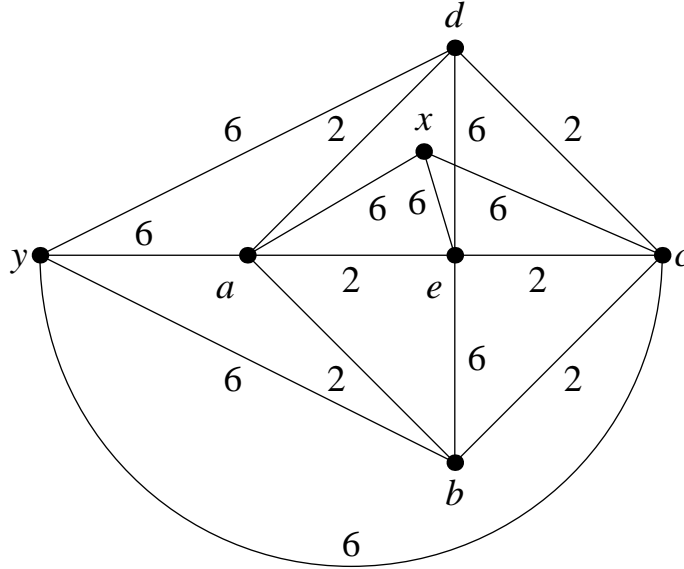


Figure 2:  $\Gamma(W, S)$

**Proposition 18** Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a visual graph of groups decomposition for  $(W, S)$  and  $V \subset S$  a vertex of  $\Psi$ . If  $A \subset V$  and  $\langle A \rangle \in \mathcal{C}(\Psi, \langle V \rangle)$ , then there exists  $B \subset V$  such that  $B$  separates  $V$  in  $\Gamma(W, S)$ ,  $\langle B \rangle \in M(\Psi, \langle V \rangle)$  and  $E(B) \subset E(A)$ .

**Proof:** If  $\langle A \rangle \in M(\Psi, \langle V \rangle)$ , let  $G = \langle A \rangle$ . Otherwise, let  $G$  be a minimum rank element of  $\mathcal{C}(\Psi, \langle V \rangle)$  such that  $G \cap \langle A \rangle$  has infinite index in  $\langle A \rangle$  and

finite index in  $G$ . In any case,  $G \in M(\Psi, \langle V \rangle)$ . By Lemma 15, there exists  $B \subset V$  and  $v \in \langle V \rangle$  such that  $B$  separates  $V$  in  $\Gamma$ ,  $v\langle B \rangle v^{-1} \subset G$  and  $\langle B \rangle \in M(\Psi, \langle V \rangle)$ . Hence  $\text{Rank}(\langle B \rangle) = \text{Rank}(G) \leq \text{Rank}(\langle A \rangle)$  and  $\langle A \rangle \cap v\langle B \rangle v^{-1}$  has finite index in  $v\langle B \rangle v^{-1}$ .

By lemma 11,  $\langle A \rangle \cap v\langle B \rangle v^{-1} = a\langle K \rangle a^{-1}$ , for some  $a \in \langle A \rangle$  and  $K \subset A$ . Hence  $E(K) \subset E(A)$ . By lemma 11,  $\langle B \rangle \cap v^{-1}\langle A \rangle v = b\langle K' \rangle b^{-1}$  for some  $b \in \langle B \rangle$  and  $K' \subset B$  where  $K$  and  $K'$  are conjugate. Hence by lemma 16  $E(K) = E(K')$ . As  $b\langle K' \rangle b^{-1}$  has finite index in  $\langle B \rangle$ ,  $E(K') = E(B)$ . Hence  $E(B) \subset E(A)$ .  $\square$

We can now easily recognize  $V$ -separating visual subgroups in either  $\mathcal{C}(\Psi, \langle V \rangle)$  or  $M(\Psi, \langle V \rangle)$ .

**Corollary 19** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a reduced visual graph of groups decomposition for  $(W, S)$ ,  $V \subset S$  is a vertex group of  $\Psi$ ,  $A \subset V$  such that  $\langle A \rangle$  splits  $\langle V \rangle$  non-trivially and visually compatible with  $\Psi$ . Then  $\langle A \rangle \in M(\Psi, \langle V \rangle)$  iff  $\langle A \rangle$  is virtually abelian, and there is no  $B \subset V$  such that  $\langle B \rangle$  is virtually abelian,  $\langle B \rangle$  splits  $\langle V \rangle$  visually compatible with  $\Psi$ , and  $E(B)$  is a proper subset of  $E(A)$ .  $\square$*

For  $(W, S)$  a Coxeter system,  $\Psi$  a reduced visual graph of groups decomposition for  $(W, S)$ ,  $V \subset S$  such that  $V$  is a vertex of  $\Psi$ , and  $A \subset V$  such that  $A$  separates  $V$  in  $\Gamma(W, S)$  and  $\langle A \rangle \in M(\Psi, \langle V \rangle)$ , we have not ruled out the possibility that there is  $x \in E(A)$  such that  $A - \{x\}$  separates  $V$  in  $\Gamma$ . It may be that  $x \in E$  for  $\langle E \rangle$  the group of an edge adjacent to  $V$  in  $\Psi$  and that in  $\Gamma$ ,  $A - \{x\}$  separates  $x$  from some other point of  $E$ , so that the visual splitting of  $W$  over  $\langle A - \{x\} \rangle$  is not compatible with  $\Psi$  (and so the minimality of  $\langle A \rangle$  is not violated). In our main applications,  $\Psi$  will be an  $n^{\text{th}}$ -stage JSJ-decomposition and we will show there is no  $B \subset V$  such that  $\langle B \rangle$  is virtually abelian and  $B$  separates  $E$  in  $\Gamma$  for  $\langle E \rangle$  the group of an edge of  $\Psi$  adjacent to  $V$ . But, until that point of the paper is reached, we add a restriction to the statements of some of our results in order to deal with this contingency.

**Lemma 20** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a reduced visual graph of groups decomposition for  $(W, S)$ ,  $V \subset S$  is a vertex of  $\Psi$ ,  $A \subset V$  separates  $V$  in  $\Gamma(W, S)$ ,  $\langle A \rangle$  is virtually abelian, and there is no  $x \in E(A)$  such that  $A - \{x\}$  separates  $V$  in  $\Gamma(W, S)$ . If  $K$  is a component of  $\Gamma - A$  which intersects  $V$  non-trivially, then for each  $a \in E(A)$  there is an edge from  $K$  to  $a$ .*

**Proof:** Otherwise,  $A - \{a\}$  separates  $V$  in  $\Gamma$ .  $\square$

**Lemma 21** *Suppose  $\Gamma$  is a graph with vertex set  $S$ , and  $A \subset S$  separates  $\{b_1, b_2\}$  in  $\Gamma$ . If  $b \in S$  is adjacent to both  $b_1$  and  $b_2$  in  $\Gamma$ , then  $b \in A$ .  $\square$*

**Lemma 22** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a reduced visual graph of groups decomposition for  $(W, S)$ ,  $V \subset S$  is a vertex of  $\Psi$ ,  $A \subset S$  such that  $\langle A \rangle$  is virtually abelian, and  $B \subset V$  such that  $\langle B \rangle$  is virtually abelian,  $B$  separates  $V$  in  $\Gamma(W, S)$  and there is no  $x \in E(B)$  such that  $B - \{x\}$  separates  $V$  in  $\Gamma(W, S)$ . If  $A$  separates  $B$  in  $\Gamma$  then  $B$  separates  $A$  in  $\Gamma$ . In particular, (by lemma 21)  $A - \{a_1, a_2\} = B - \{b_1, b_2\}$  for  $a_1, a_2$  unrelated elements of  $A$  and  $b_1, b_2$  unrelated elements of  $B$ . (So  $\text{Rank}(A) = \text{Rank}(B)$ .) Furthermore, if  $A \subset V$ ,  $\langle A \rangle \in \mathcal{C}(\Psi, \langle V \rangle)$ ,  $\langle B \rangle \in M(\Psi, \langle V \rangle)$ , and for each  $D \subset V$  such that  $\langle D \rangle \in M(\Psi, \langle V \rangle)$  there is no  $x \in E(D)$  such that  $D - \{x\}$  separates  $V$  in  $\Gamma(W, S)$  then  $\langle A \rangle \in M(\Psi, \langle V \rangle)$ .*

**Proof:** Write  $B = \{b_1, b_2\} \cup M$ , where  $A$  separates  $b_1$  and  $b_2$  in  $\Gamma$  and  $\langle M \rangle$  commutes with  $\{b_1, b_2\}$ . By lemma 21,  $M \subset A$  and so  $A \cap B = M$ . If  $B$  does not separate  $A$ , then  $A - M \subset K$  for  $K$  a component of  $\Gamma - B$ . For  $i \in \{1, 2\}$ , let  $K_i$  be distinct components of  $\Gamma - B$  containing  $t_i \in V$ . Assume  $K_1 \neq K_2$ . Then  $t_1 \notin A$ . We may assume  $t_1$  is in a component of  $\Gamma - A$  not containing  $b_2$ . By hypothesis,  $\{b_1\} \cup M$  does not separate  $V$  in  $\Gamma$ . Choose a shortest path from  $t_1$  to  $t_2$  avoiding  $\{b_1\} \cup M$ . Then this path passes through  $b_2$  (exactly once). As  $A$  separates  $b_2$  and  $t_1$ , we let  $s$  be the first vertex of  $A$  in our path. Then  $s \in A - M \subset K$  and we have connected  $t_1 \in K_1$  to  $A - M \subset K$  in  $\Gamma - B$ , which is nonsense. By lemma 21,  $A - \{a_1, a_2\} = B - \{b_1, b_2\} \equiv M$ .

It remains to show that if  $A \subset V$ ,  $\langle A \rangle \in \mathcal{C}(\Psi, \langle V \rangle)$  and  $\langle B \rangle \in M(\Psi, \langle V \rangle)$  then  $\langle A \rangle \in M(\Psi, \langle V \rangle)$ . Otherwise, proposition 18 implies there is  $D \subset V$  such that  $D$  separates  $V$  in  $\Gamma$ ,  $\langle D \rangle \in M(\Psi, \langle V \rangle)$ ,  $E(D) \subset E(A) (= \{a_1, a_2\} \cup E(M))$  and  $\text{Rank}(\langle D \rangle) < \text{Rank}(\langle A \rangle)$ . If  $\{a_1, a_2\} \subset D$ , then as  $B$  separates  $a_1$  and  $a_2$ , the argument above shows  $D - \{a_1, a_2\} = B - \{b'_1, b'_2\}$ . But then  $\text{Rank}(\langle D \rangle) = \text{Rank}(\langle B \rangle) = \text{Rank}(\langle A \rangle)$  which is nonsense. If  $a_1 \notin D$ , then  $E(D) \subset M$ . This is impossible as  $\langle B \rangle \in M(\Psi, \langle V \rangle)$  and  $\langle B \rangle = \langle b_1, b_2 \rangle \times \langle M \rangle$ . Similarly for  $a_2$ .  $\square$

If  $\langle A \rangle$  and  $\langle B \rangle$  are elements of  $M(\Psi, \langle V \rangle)$  and  $A$  and  $B$  separate one another in  $\Gamma$ , we say  $\langle A \rangle$  and  $\langle B \rangle$  *cross* or are *crossing* in  $M(\Psi, \langle V \rangle)$ .

**Proposition 23** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a reduced visual graph of groups decomposition for  $(W, S)$ ,  $V \subset S$  is a vertex of  $\Psi$ ,  $A \subset S$  such that  $\langle A \rangle$  is virtually abelian, and  $B \subset V$  such that  $\langle B \rangle$  is virtually abelian,  $B$  separates  $V$  in  $\Gamma(W, S)$  and there is no  $x \in E(B)$  such that  $B - \{x\}$  separates  $V$  in  $\Gamma(W, S)$ . If  $A$  separates  $B$  in  $\Gamma(W, S)$  then  $\Gamma(W, S) - B$  has exactly 2-components which intersect  $V$  non-trivially.*

**Proof:** By lemma 22, we may assume  $A = \{a_1, a_2\} \cup M$  and  $B = \{b_1, b_2\} \cup M$ , where  $a_1$  and  $a_2$  are unrelated and separated by  $B$ ,  $b_1$  and  $b_2$  are unrelated and separated by  $A$ , and  $M$  commutes with  $\{a_1, a_2, b_1, b_2\}$ . For  $i \in \{1, 2\}$ , let  $K_i$  be the component of  $\Gamma - B$  containing  $a_i$ .

If  $K$  is a component of  $\Gamma - B$  which intersects  $V$  non-trivially, then by lemma 20, there is an edge connecting  $b_i$  to  $K$  for  $i \in \{1, 2\}$ . If additionally,  $K \neq K_i$  for  $i \in \{1, 2\}$ , then  $K \cap A = \emptyset$ , and so  $K \cup \{b_1, b_2\}$  is a connected subset of  $\Gamma - A$ . This is impossible since  $A$  separates  $b_1$  and  $b_2$  in  $\Gamma$ .  $\square$

**Lemma 24** *Suppose  $(W, S)$  is a Coxeter system and  $\Lambda$  is a graph of groups decomposition of  $W$ . If  $A \subset S$ ,  $\langle A \rangle$  is virtually abelian and for any unrelated  $x, y$  in  $A$ ,  $\{x, y\}$  stabilizes a vertex of  $T_\Lambda$  (the Bass-Serre tree for  $\Lambda$ ), then  $A$  stabilizes a vertex of  $T_\Lambda$ . In particular, if  $\Psi$  is the visual graph of groups for  $\Lambda$  given by the construction for theorem 7, then  $A \subset V$  for  $\langle V \rangle$  a vertex group of  $\Psi$ .*

**Proof:** The group  $\langle A \rangle$  decomposes as  $\langle a_1, b_1 \rangle \times \cdots \times \langle a_n, b_n \rangle \times \langle F \rangle$  where  $A = \{a_1, b_1, \dots, a_n, b_n\} \cup F$ ,  $m(a_i, b_i) = \infty$  and  $F$  generates a complete subdiagram of  $\Gamma(W, S)$ . If the statement of the lemma fails, assume  $n$  is minimal among all counterexamples. Note that  $n > 0$  since  $F$  is FA (see [11]). By the minimality of  $n$ ,  $\langle a_1, \dots, a_n, b_2, \dots, b_n, F \rangle$  stabilizes a vertex  $V_1$  of  $T_\Lambda$ ,  $\langle a_2, \dots, a_n, b_1, \dots, b_n, F \rangle$  stabilizes  $V_2$  and  $\langle a_1, b_1 \rangle$  stabilizes  $V_3$ . As  $T_\Lambda$  is a tree, there is a vertex  $V$  of  $T_\Lambda$  common to the three geodesics connecting pairs in  $\{V_1, V_2, V_3\}$ , and  $A$  stabilizes  $V$ .  $\square$

## 5 Weak $M(\Psi, \langle V \rangle)$ -JSD Decompositions

If  $(W, S)$  is a Coxeter system,  $\Psi$  is a reduced visual graph of groups decomposition for  $(W, S)$  and  $V \subset S$  is a vertex of  $\Psi$ , then a reduced  $\Psi$ -compatible graph of groups decomposition,  $\Lambda$ , of  $\langle V \rangle$  is *weakly  $M(\Psi, \langle V \rangle)$ -JSJ* if

1. each edge group of  $\Lambda$  is in  $M(\Psi, \langle V \rangle)$  and
2. each element of  $M(\Psi, \langle V \rangle)$  is a subgroup of a conjugate of a vertex group of  $\Lambda$ .

A reduced  $\Psi$ -compatible visual graph of groups decomposition  $\Psi_1$  of  $\langle V \rangle$  looks weakly  $M(\Psi, \langle V \rangle)$ -JSJ if

1. each edge group of  $\Psi_1$  is in  $M(\Psi, \langle V \rangle)$  and
2. no edge group  $\langle E \rangle$  ( $E \subset V$ ) of  $\Psi_1$  is crossing.

At this point we consider decompositions that are precursors to the  $n^{\text{th}}$ -stage JSJ-decompositions. For a finitely generated Coxeter system  $(W, S)$  we say a visual graph of groups decomposition  $\Psi$  is *JSJ-amenable* if  $\Psi$  is reduced and for any vertex  $V \subset S$  of  $\Psi$  and any  $E \subset S$  such that  $\langle E \rangle$  is the group of an edge adjacent to  $V$ , there is no  $A \subset V$  such that  $\langle A \rangle$  is virtually abelian and  $A$  separates  $E$  in  $\Gamma$ . In particular, if  $\Psi$  is JSJ-amenable,  $V \subset S$  is a vertex of  $\Psi$ ,  $A \subset V$  separates  $V$  in  $\Gamma$  and  $\langle A \rangle \in M(\Psi, \langle V \rangle)$  then there is no  $x \in E(A)$  such that  $A - \{x\}$  separates  $V$  in  $\Gamma$ . (This remark should be compared with the one following corollary 19). In section 7 (proposition 34) we show that  $n^{\text{th}}$ -stage JSJ-decompositions are JSJ-amenable.

**Proposition 25** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a visual graph of groups decomposition for  $(W, S)$  that is JSJ-amenable and  $V \subset S$  is a vertex of  $\Psi$ . A reduced visual  $\Psi$ -compatible graph of groups decomposition  $\Psi_1$  of  $\langle V \rangle$  looks weakly  $M(\Psi, \langle V \rangle)$ -JSJ if and only if it is weakly  $M(\Psi, \langle V \rangle)$ -JSJ.*

**Proof:** Assume  $\Psi_1$  is weakly  $M(\Psi, \langle V \rangle)$ -JSJ. If  $\Psi_1$  does not look weakly  $M(\Psi, \langle V \rangle)$ -JSJ, then there is  $E \subset V$  such that  $\langle E \rangle$  is the edge group of an edge  $E'$  of  $\Psi_1$  and  $T \subset V$  such that  $\langle T \rangle \in M(\Psi, \langle V \rangle)$  and  $\langle E \rangle$  crosses  $\langle T \rangle$  in  $M(\Psi, \langle V \rangle)$ . Assume  $E$  separates elements  $t_1$  and  $t_2$  of  $T$  in  $\Gamma$ . By lemma 22,  $E = \{e_1, e_2\} \cup N$  and  $T = \{t_1, t_2\} \cup N$  where  $e_1$  and  $e_2$  are unrelated,  $t_1$  and  $t_2$  are unrelated and  $N$  commutes with  $\{e_1, e_2, t_1, t_2\}$ . Since  $\Psi_1$  is weakly  $M(\Psi, \langle V \rangle)$ -JSJ, there is a vertex  $U \subset V$  of  $\Psi_1$  such that  $\langle T \rangle$  is conjugate to a subgroup of  $\langle U \rangle$ . By proposition 17,  $\{t_1, t_2\} \subset U$ . As  $\Psi_1$  is a tree, we may assume  $E'$  is an edge of  $\Psi_1$  adjacent to  $U$ . Let  $Q$  be the vertex of  $E'$  opposite  $U$ . Let  $\Psi'$  be the graph of groups decomposition obtained from  $\Psi$  by replacing  $\langle V \rangle$  by  $\Psi_1$ . Since  $\Psi_1$  is reduced, there exists  $x \in Q - E$ .

By proposition 23,  $\Gamma - E$  has exactly two components which intersect  $V$  non-trivially, one containing  $t_1$  and the other containing  $t_2$ . Hence  $x$  can be connected to  $t_1$  or  $t_2$  by a path in  $\Gamma - E$ . This is impossible as  $E'$  separates  $U$  and  $Q$  in  $\Psi'$  and so by lemma 9,  $E$  separates  $U - E$  and  $Q - E$  in  $\Gamma$ .

Suppose  $\Psi_1$  looks weakly  $M(\Psi, \langle V \rangle)$ -JSJ and  $B \in M(\Psi, \langle V \rangle)$ . By proposition 18, there is  $A \subset V$  such that  $\langle A \rangle \in M(\Psi, \langle V \rangle)$  and some conjugate of  $\langle A \rangle$  is a subgroup of (finite index in)  $B$ .

Suppose  $x$  and  $y$  are elements of  $A$  and there is no vertex group of  $\Psi_1$  containing  $\{x, y\}$ . Then  $x$  and  $y$  are separated by  $E$  in  $\Gamma(W, S)$  for  $E$  an edge of  $\Psi_1$ . This is impossible as  $\Psi_1$  looks weakly  $M(\Psi, \langle V \rangle)$ -JSJ. We conclude that  $\{x, y\} \subset U$  for some  $U \subset V$  a vertex of  $\Psi_1$ .

By lemma 24,  $A$  is a subgroup of a conjugate of a vertex group of  $\Psi_1$ . Let  $T$  be the Bass-Serre tree for  $\Psi_1$  and  $U$  a vertex of  $T$  stabilized by  $\langle A \rangle$ . Let  $B'$  be a conjugate of  $B$  such that  $\langle A \rangle$  has finite index in  $B'$ . If the cosets of  $\langle A \rangle$  in  $B'$  are  $\langle A \rangle, b_1 \langle A \rangle, \dots, b_n \langle A \rangle$ , then the orbit of  $B'U$  in  $T$  is  $U, b_1 U, \dots, b_n U$ . By corollary 4.8 of [5],  $B'$  stabilizes some vertex of  $T$ . Equivalently,  $B$  is a subgroup of a conjugate of a vertex group of  $\Psi_1$ , and hence  $\Psi_1$  is weakly  $M(\Psi, \langle V \rangle)$ -JSJ.  $\square$

**Lemma 26** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  a reduced visual graph of groups decomposition for  $(W, S)$ ,  $V \subset S$  a vertex of  $\Psi$ ,  $\Lambda$  a reduced  $\Psi$ -compatible graph of groups decomposition of  $\langle V \rangle$ ,  $\Psi_1$  the reduced visual decomposition for  $\Lambda$  from the construction for theorem 7,  $E \subset V$  such that  $\langle E \rangle$  is an edge group of  $\Psi_1$ ,  $T$  the Bass-Serre tree for  $\Lambda$  and  $E'$  the edge of  $T$  such that  $E = \{v \in V : v \text{ stabilizes } E'\}$ . If  $\{x, y\} \subset V - E$  and  $x$  (respectively  $y$ ) stabilizes the vertex  $X$  (respectively  $Y$ ) of  $T$  where  $X$  and  $Y$  are on different sides of  $E'$ , then  $x$  and  $y$  are in different components of  $\Gamma - E$ .*

**Proof:** If  $\Phi$  is obtained from  $\Psi$  by replacing  $V$  by  $\Lambda$ , then  $T$  is a subtree of the Bass-Serre tree for  $\Phi$ . Hence,  $E$  is an edge of  $\Psi_\Phi$  the visual decomposition for  $\Phi$  given by the construction for theorem 7. Now apply lemma 9.  $\square$

**Proposition 27** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a visual graph of groups decomposition for  $(W, S)$  that is JSJ-amenable,  $V \subset S$  is a vertex of  $\Psi$ ,  $\Lambda$  is a reduced graph of groups decomposition of  $\langle V \rangle$  with edge groups in  $M(\Psi, \langle V \rangle)$ , and  $\Psi_1$  is the reduced visual decomposition for  $\Lambda$  given by the construction for theorem 7. Then  $\Psi_1$  is weakly  $M(\Psi, \langle V \rangle)$ -JSJ if and only if  $\Lambda$  is weakly  $M(\Psi, \langle V \rangle)$ -JSJ.*

**Proof:** By lemmas 3 and 15,  $\Psi_1$  and  $\Lambda$  are  $\Psi$ -compatible, and for each  $E \subset V$  such that  $\langle E \rangle$  is an edge group of  $\Psi_1$ ,  $\langle E \rangle$  is in  $M(\Psi, \langle V \rangle)$  and  $E$  separates  $V$  in  $\Gamma(W, S)$ . Assume  $\Lambda$  is weakly  $M(\Psi, \langle V \rangle)$ -JSJ. If  $\Psi_1$  is not weakly  $M(\Psi, \langle V \rangle)$ -JSJ, then proposition 25 implies  $\Psi_1$  does not look weakly  $M(\Psi, \langle V \rangle)$ -JSJ. I.e. there are  $E$  and  $B$ , subsets of  $V$  that separate  $\Gamma(W, S)$  and generate crossing members of  $M(\Psi, \langle V \rangle)$ , such that  $E$  separates (in  $\Gamma$ ) elements  $b_1$  and  $b_2$  of  $B$ , and  $\langle E \rangle$  is an edge group of  $\Psi_1$ . Let  $T$  be the Bass-Serre tree for  $\Lambda$ . The construction of visual decompositions for theorem 7 implies there is an edge  $E'$  of  $T$  such that  $E = \{v \in V : v \text{ stabilizes } E'\}$  and since  $\Psi_1$  is reduced, we may assume there are elements  $x$  and  $y$  of  $V - E$  that stabilize vertices  $X$  and  $Y$  (respectively) of  $T$  on opposite sides of  $E'$ . Since  $\Lambda$  is weakly  $M(\Psi, \langle V \rangle)$ -JSJ, there is a vertex  $U$  of  $T$  such that  $B$  stabilizes  $U$ . By proposition 23,  $\Gamma - E$  has exactly two components which intersect  $V$  non-trivially, one containing  $b_1$  and the other containing  $b_2$ . There is an element  $z$  of  $V - E$  that stabilizes a vertex of  $T$  on the side of  $E'$  opposite  $U$ . But by lemma 26,  $z$  cannot be in the same component of  $\Gamma - E$  as  $b_1$  or  $b_2$ , which is nonsense. The converse is trivial.  $\square$

## 6 $M(\Psi, \langle V \rangle)$ -JSJ Decompositions

Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a reduced visual graph of groups decomposition for  $(W, S)$ , and  $V \subset S$  is a vertex of  $\Psi$ . We define, a weakly  $M(\Psi, \langle V \rangle)$ -JSJ decomposition  $\Lambda$ , to be  $M(\Psi, \langle V \rangle)$ -JSJ if for any vertex group  $U$  of  $\Lambda$  and non-trivial  $\Lambda$ -compatible splitting of  $U$  over an  $M(\Psi, \langle V \rangle)$  subgroup, the resulting (reduced) decomposition of  $\Lambda$  is not weakly  $M(\Psi, \langle V \rangle)$ -JSJ.

Suppose  $\Psi_1$  is a visual weak  $M(\Psi, \langle V \rangle)$ -JSJ decomposition,  $E \subset V$  such that  $\langle E \rangle$  is a non-crossing element of  $M(\Psi, \langle V \rangle)$ , and  $E$  separates  $U$  in  $\Gamma(W, S)$  for  $U$  a vertex of  $\Psi_1$ . Since  $\langle E \rangle$  is non-crossing, this splitting is  $\Psi_1$ -compatible, visual and non-trivial. By lemma 3, the resulting decomposition of  $\Psi_1$  is compatible with  $\Psi$ . We say a visual weakly  $M(\Psi, \langle V \rangle)$ -JSJ decomposition  $\Psi_1$  *looks*  $M(\Psi, \langle V \rangle)$ -JSJ if for any  $E \subset V$  such that  $\langle E \rangle$  is a non-crossing member of  $M(\Psi, \langle V \rangle)$  and vertex  $U$  of  $\Psi_1$  such that  $E \subset U \subset V$ ,  $E$  does not separate  $U$  in  $\Gamma$ .

The next proposition implies the existence of visual  $M(\Psi, \langle V \rangle)$ -JSJ decompositions for a given JSJ-amenable graph of groups decomposition  $\Psi$ .

**Proposition 28** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a visual graph of groups decomposition of  $(W, S)$  that is JSJ-amenable, and  $V \subset S$  is a vertex of  $\Psi$ . A visual weak  $M(\Psi, \langle V \rangle)$ -JSJ decomposition  $\Psi_1$  of  $\langle V \rangle$ , looks  $M(\Psi, \langle V \rangle)$ -JSJ if and only if  $\Psi_1$  is  $M(\Psi, \langle V \rangle)$ -JSJ.*

**Proof:** Suppose  $\Psi_1$  looks  $M(\Psi, \langle V \rangle)$ -JSJ, but is not  $M(\Psi, \langle V \rangle)$ -JSJ. Then there is  $D \in M(\Psi, \langle V \rangle)$  and a vertex  $U \subset V$  of  $\Psi_1$  such that  $\langle U \rangle$  splits non-trivially and  $\Psi_1$ -compatibly as  $A *_D B$ , and the resulting reduced decomposition  $\Lambda$  (of  $\langle V \rangle$ ), is weakly  $M(\Psi, \langle V \rangle)$ -JSJ. Let  $\Psi'$  be the reduced visual decomposition for  $A *_D B (= \langle U \rangle)$  given by the construction for theorem 7. As no conjugate of  $\langle U \rangle$  is contained in  $A$  or  $B$ ,  $\Psi'$  has more than one vertex. Any visual subgroup of  $\langle U \rangle$  contained in a conjugate of  $A$  or  $B$  is contained in a vertex group of  $\Psi'$  by the construction for theorem 7. Hence  $\Psi'$  is compatible with  $\Psi_1$ . Let  $\Psi'_1$  be obtained from  $\Psi_1$  by replacing  $\langle U \rangle$  by  $\Psi'$ , and reducing. Then  $\Psi'_1$  is a reduced visual graph of groups decomposition for  $\Lambda$  as given by the construction for theorem 7. By proposition 27,  $\Psi'_1$  is weakly  $M(\Psi, \langle V \rangle)$ -JSJ and so  $\Psi_1$  did not look  $M(\Psi, \langle V \rangle)$ -JSJ to begin with. The converse is trivial.  $\square$

**Theorem 29** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a visual graph of groups decomposition of  $(W, S)$  that is JSJ-amenable, and  $V \subset S$  is a vertex of  $\Psi$ . If  $\Lambda$  is a reduced  $M(\Psi, \langle V \rangle)$ -JSJ graph of groups decomposition of  $\langle V \rangle$ , and  $\Psi_1$  is the reduced visual decomposition derived from  $\Lambda$  by the construction for theorem 7, then*

1.  $\Psi_1$  is an  $M(\Psi, \langle V \rangle)$ -JSJ decomposition.
2. There is a (unique) bijection  $\alpha$  of the vertices of  $\Lambda$  to the vertices of  $\Psi_1$  such that for each vertex  $U$  of  $\Lambda$ ,  $\Lambda(U)$  is conjugate to  $\Psi(\alpha(U))$ .
3. Each edge group of  $\Lambda$  is conjugate to a special subgroup of  $\langle V \rangle$ .

**Proof:** The decomposition  $\Psi_1$  is weakly  $M(\Psi, \langle V \rangle)$ -JSJ by Proposition 27. Suppose  $U$  is a vertex of  $\Lambda$  with vertex group  $A = \Lambda(U)$ . We wish to show that  $A$  is a subgroup of a conjugate of a vertex group of  $\Psi_1$ . Otherwise, the action of  $A$  on  $T$ , the Bass-Serre tree for  $\Psi_1$ , defines a non-trivial reduced graph of groups decomposition  $\Phi$  of  $A$  such that each edge group of  $\Phi$  is a subgroup of a conjugate of an  $(M(\Psi, \langle V \rangle))$  edge group of  $\Psi_1$ . If  $A$  does not stabilize a vertex of  $T$ , then  $\Phi$  is nontrivial. If  $C$  is an edge group of

$\Lambda$  incident with  $A$ , then  $C$  contains a conjugate of an edge group  $Q$  of  $\Psi_1$ . Since  $C$  and  $Q$  are in  $M(\Psi, \langle V \rangle)$ , a conjugate of  $Q$  is of finite index in  $C$  and at the same time stabilizes an edge of  $T$ . But then the orbit of this edge in  $T$  under the action of  $C$  is finite. By Corollary 4.8 of [5],  $C$  stabilizes some vertex of  $T$  and so is contained in a conjugate of a vertex group of  $\Phi$ . Hence  $\Phi$  is compatible with  $\Lambda$ . We also require  $\Phi$  to be compatible with  $\Psi$ . Suppose  $D \subset V$  is an edge of  $\Psi$  adjacent to  $V$ , such that some conjugate of  $\langle D \rangle$  is a subgroup of  $A$ . Then  $D \subset Q$  for  $Q$  a vertex of  $\Psi_1$ . Hence  $\langle D \rangle$  stabilizes a vertex group of  $T$  and so is a subgroup of a conjugate of a vertex group of  $\Phi$ , as required. Replacing  $A$  in  $\Lambda$  by this decomposition and reducing gives a  $\Psi$ -compatible reduced graph of groups decomposition  $\Lambda'$  of  $\langle V \rangle$  with  $M(\Psi, \langle V \rangle)$  edge groups. By hypothesis,  $\Lambda'$  is not weakly  $M(\Psi, \langle V \rangle)$ -JSJ. Hence there exists  $B$ , an  $M(\Psi, \langle V \rangle)$  subgroup of  $\langle V \rangle$  such that  $\langle V \rangle$  splits non-trivially over  $B$  and  $B$  is not a subgroup of a conjugate of a vertex group of  $\Lambda'$ . Since  $\Lambda$  is weakly  $M(\Psi, \langle V \rangle)$ -JSJ,  $B$  is a subgroup of a conjugate of  $A$ . But, since  $\Psi_1$  is weakly  $M(\Psi, \langle V \rangle)$ -JSJ,  $B$  is a subgroup of a conjugate of a vertex group of  $\Psi_1$  and hence of a vertex group of  $\Phi$  and of  $\Lambda'$ . We conclude that  $A$  is a subgroup of a conjugate of a vertex group of  $\Psi_1$ , as desired.

If  $A (= \Lambda(U))$  is a subgroup of a conjugate of  $\Psi_1(U') (\equiv \langle U' \rangle)$  for  $U'$  a vertex of  $\Psi_1$ , then, since  $\Psi_1(U')$  is a subgroup of a conjugate of a vertex group of  $\Lambda$ ,  $A$  is a subgroup of a conjugate of a vertex group of  $\Lambda$ . By lemma 2 the vertex group  $A$  at  $U$  is a subgroup of a conjugate of a vertex group at  $U''$  only if  $U = U''$  and (since  $\Lambda$  is a tree) the conjugate is by an element of  $A$ . But then  $A$  is conjugate to  $\Psi_1(U')$ . Since no vertex group of  $\Psi_1$  is contained in a conjugate of another,  $U'$  is uniquely determined, and we set  $\alpha(U) = U'$ . Since each vertex group  $\Psi_1(U')$  is contained in a conjugate of some  $\Lambda(U)$  which is in turn conjugate to  $\Psi_1(\alpha(U))$  we must have  $U' = \alpha(U)$  and each  $U'$  is in the image of  $\alpha$ .

If  $\Psi_1$  is not  $M(\Psi, \langle V \rangle)$ -JSJ, then it does not look  $M(\Psi, \langle V \rangle)$ -JSJ and some vertex group  $W_1$  of  $\Psi_1$  visually splits nontrivially and  $\Psi_1$ -compatibly over an  $M(\Psi, \langle V \rangle)$  visual subgroup  $U_1$  to give a  $\Psi$ -compatible visual graph of groups decomposition  $\Phi$  of  $\langle V \rangle$  with  $U_1$  an edge group. Now  $W_1$  is conjugate to a vertex group  $A$  of  $\Lambda$ . As a subgroup of  $\langle V \rangle$ ,  $A$  acts on the Bass-Serre tree  $T'$  for  $\Phi$ , but  $A$  cannot stabilize a vertex of  $T'$ , otherwise  $W_1$  stabilizes a vertex of  $T'$  and we assumed  $W_1$  split nontrivially. As above this contradicts the assumption that  $\Lambda$  is  $M(\Psi, \langle V \rangle)$ -JSJ, implying instead that  $\Psi_1$  is  $M(\Psi, \langle V \rangle)$ -JSJ.

Since  $\Lambda$  is a tree, we can take each edge group of  $\Lambda$  as contained in its

endpoint vertex groups taken as subgroups of  $\langle V \rangle$ . Hence each edge group is simply the intersection of its adjacent vertex groups (up to conjugation). Since vertex groups of  $\Lambda$  correspond to conjugates of vertex groups in  $\Psi_1$ , their intersection is conjugate to a visual subgroup by Lemma 11.  $\square$

Theorem 29 shows that all  $M(\Psi, \langle V \rangle)$ -JSJ decompositions for Coxeter groups are basically visual. The next collection of lemmas lead to a proof of the uniqueness of  $M(\Psi, \langle V \rangle)$ -JSJ decompositions for Coxeter groups.

**Lemma 30** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a visual graph of groups decomposition of  $(W, S)$  that is JSJ-amenable,  $V \subset S$  is a vertex of  $\Psi$  and  $\Psi_1$  is a visual  $M(\Psi, \langle V \rangle)$ -JSJ decomposition for  $\langle V \rangle$ . If  $D \subset V$  is such that  $D$  separates  $V$  in  $\Gamma(W, S)$  and  $\langle D \rangle \in M(\Psi, \langle V \rangle)$ , then there is  $Q \subset V$  a vertex of  $\Psi_1$  such that  $D \subset Q$ . Furthermore, if  $\langle D \rangle$  is non-crossing in  $M(\Psi, \langle V \rangle)$ , then  $E(D)$  is a subset of an edge group of  $\Psi_1$ .*

**Proof:** By definition some conjugate of  $D$  is a subset of  $\langle Q \rangle$  for  $Q$  a vertex of  $\Psi_1$ , but we require more. By proposition 17,  $E(D) \subset Q$ . We have  $D = E(D) \cup N$  where  $N \subset V$  generates a finite subgroup of  $\langle V \rangle$  which commutes with  $E(D)$ . As  $N$  determines a complete subdiagram of  $\Gamma(\langle V \rangle, V)$ ,  $N$  is a subset of some vertex of  $\Psi_1$ . Let  $D'$  be a largest subset of  $D$  such that  $N \subset D'$ ,  $D' \subset Q_1 \subset V$  and  $Q_1$  is a vertex of  $\Psi_1$ . If  $D' \neq D$ , let  $a \in D - D'$ . Observe that  $\{a\} \cup N$  generates a finite group and so there exists  $Q_2 (\subset V)$  a vertex of  $\Psi_1$  such that  $\{a\} \cup N \subset Q_2$ . If  $U$  is a vertex common to the three  $\Psi_1$ -geodesics connecting pairs in  $\{Q, Q_1, Q_2\}$ , then  $\{a\} \cup D' \subset U$  contrary to the definition of  $D'$ . We conclude  $D' = D$  as required.

The second part of the lemma is not used in the rest of the paper and is tedious to prove. The reader may wish to skip this argument on a first reading.

Assume  $\langle D \rangle$  is non-crossing in  $M(\Psi, \langle V \rangle)$  and  $Q$  is a vertex of  $\Psi_1$  with  $D \subset Q \subset V$ . Assume  $E(D) \neq \emptyset$ . If  $Q'$  is a vertex of  $\Psi_1$ , distinct from  $Q$  and  $E(D) \subset Q'$  then for any  $F \subset V$  such that  $\langle F \rangle$  is the edge group of an edge on the  $\Psi_1$ -geodesic connecting  $Q$  and  $Q'$ ,  $E(D) \subset F$ . Hence we may assume:

(\*)  $Q$  is the only vertex of  $\Psi_1$  such that  $E(D) \subset Q$ .

Even as  $D$  separates  $V$  in  $\Gamma$ ,  $D$  does not separate  $Q$  in  $\Gamma$  (otherwise, since  $D$  is non-crossing and does not separate  $E$  (for  $\langle E \rangle$  an edge group of  $\Psi$ ) in  $\Gamma$ ,  $\langle Q \rangle$  splits non-trivially and compatibly with  $\Psi_1$  and  $\Psi$  over  $D$  - contrary to our JSJ assumption on  $\Psi_1$ ). Hence  $Q - D \subset C$  for  $C$  a component of  $\Gamma - D$ .

Next we show that if  $C'$  is a component of  $\Gamma - D$  other than  $C$  such that  $C'$  intersects  $V$  non-trivially, and  $x \in E(D)$  then there is a vertex  $V_{x,C'}$  of  $\Psi_1$  such that  $x \in V_{x,C'}$  and  $V_{x,C'} \cap C' \neq \emptyset$ . First observe that  $C' \cap Q = \emptyset$ . By lemma 20, for each  $x \in E(D)$  there is an edge  $[xa]$  of  $\Gamma$  such that  $a \in C'$ . Let  $x \equiv x_0, x_1, \dots, x_n$  be the consecutive vertices of a path in  $\Gamma$  from  $x$  to  $x_n \in V \cap C'$  such that for  $i > 0$ ,  $x_i \in C'$ . We may assume  $n$  is the smallest integer such that  $x_n \in V \cap C'$ . Since  $x \notin C'$ ,  $n \neq 0$ . Let  $\Psi'$  be the visual graph of groups decomposition of  $W$  obtained by replacing the vertex  $V$  of  $\Psi$  by  $\Psi_1$ . By lemma 8, for each  $i \in \{1, 2, \dots, n\}$  there is a vertex  $V_i \subset S$  of  $\Psi'$  such that  $\{x_{i-1}, x_i\} \subset V_i$ . Define  $V_0 \equiv Q$ . Since  $Q \cap C' = \emptyset$ ,  $V_i \neq Q$  for  $i > 0$ . Let  $\alpha_i$  be the  $\Psi'$ -geodesic from  $Q \equiv V_0$  to  $V_i$  and  $\beta_i$  be the  $\Psi'$ -geodesic from  $V_i$  to  $V_{i+1}$  for  $i \geq 1$ . By lemma 8, if  $B \subset S$  is a vertex of  $\beta_i$ , then  $x_i \in C' \cap B$  and so  $\beta_i$  does not pass through  $Q$ . This implies that  $\alpha_n$  can be written as a non-trivial subpath  $\tau$  of  $\alpha_1$  followed by a path  $\lambda$  where each edge of  $\lambda$  is an edge of some  $\beta_i$ . Let  $Q \equiv X_1, \dots, X_k \equiv V_n$  be the consecutive vertices of  $\alpha_n$ . If  $X_i$  is the end point of  $\tau$  then  $x_0 \in X_i$  (lemma 8 implies  $x_0$  is an element of every vertex of  $\alpha$ ). Since  $X_i$  is also the initial point of  $\lambda$ ,  $\{x_0, x_m\} \subset X_i$  for some  $m \in \{1, \dots, n\}$ . Since  $\Psi_1$  is a subtree of  $\Psi'$ , there is  $j \in \{1, \dots, k\}$  such that  $X_1, \dots, X_j$  are vertices  $\Psi_1$  and  $X_{j+1}, \dots, X_k$  are not. If  $j \geq i$ , then as  $\{x_0, x_m\} \subset X_i$ , set  $V_{x,C'} = X_i$  to finish. If  $j < i$ , then as  $\Psi_1$  is a subtree of  $\Psi'$ , any  $\Psi'$ -geodesic from  $V_n \equiv X_k$  to a vertex of  $\Psi_1$  must have initial segment with consecutive vertices  $V_n \equiv X_k, X_{k-1}, \dots, X_j$ . Hence if  $Q' \subset V$  is a vertex of  $\Psi_1$  containing  $x_n (\in V \cap C')$ , then the  $\Psi'$  geodesic from  $V_n$  to  $Q'$  passes through  $X_j$ . As  $x_n \in V_n \cap Q'$ , lemma 8 implies  $x_n \in X_j$ . Since  $j < i$ ,  $x_0 \in X_j$  and set  $V_{x,C'} = X_j$  to finish the claim.

Note that for all  $x \in E(D)$  and for all components  $C' \neq C$  of  $\Gamma - D$ ,  $V_{x,C'} \neq Q$  (as  $C' \cap Q = \emptyset$ ). For  $x \in E(D)$  and  $C' \neq C$  a component of  $\Gamma - D$ , there may be more than one possible choice for  $V_{x,C'}$ . Assume that  $E(D) \cap V_{x,C'}$  is maximal over all possible choices. I.e. if  $X$  is a vertex of  $\Psi_1$ ,  $x \in X$ , and  $X \cap C' \neq \emptyset$  then the number of elements of  $X \cap E(D)$  is less than or equal to the number of elements of  $V_{x,C'} \cap E(D)$ . By (\*),  $E(D) \not\subset V_{x,C'}$ . Let  $y \in E(D) - V_x$ , then  $Q$ ,  $V_{y,C'}$  and  $V_{x,C'}$  are distinct. There is a vertex  $U$  common to the three geodesics of  $\Psi_1$  connecting pairs in  $\{Q, V_{x,C'}, V_{y,C'}\}$  and so  $[E(D) \cap V_{x,C'}] \cup \{y\} \subset U$ . In particular,  $x \in U$ . By the maximality condition on  $V_{x,C'} \cap E(D)$ ,  $U \cap C' = \emptyset$ . Hence  $V_{x,C'} \neq U \neq V_{y,C'}$  and  $U$  separates  $V_{x,C'}$  and  $V_{y,C'}$  in  $\Psi_1$ . If  $\Psi'$  is the graph of groups decomposition of  $W$  obtained from  $\Psi$  by replacing  $V$  by  $\Psi_1$ , then  $\Psi_1$  is a subtree of the tree  $\Psi'$  and  $U$  separates  $V_{x,C'}$  and  $V_{y,C'}$  in  $\Psi'$ . Suppose  $\alpha$  is an edge path in  $C'$  (with

consecutive vertices  $c_0, \dots, c_n$ ) connecting  $c_0 \in C' \cap V_{x,C'}$  to  $c_n \in C' \cap V_{y,C'}$ . Let  $C_0 = V_{x,C'}$ ,  $C_i$  be a vertex of  $\Psi'$  such that  $C_i$  contains  $\{c_{i-1}, c_i\}$  and  $C_{n+1} = V_{y,C'}$ . If  $X$  is a vertex of the  $\Psi'$ -geodesic from  $C_i$  to  $C_{i+1}$ , then lemma 8 implies  $X$  contains  $c_i \in C'$ . Hence,  $\alpha$  defines a path in  $\Psi'$  from  $V_{x,C'}$  to  $V_{y,C'}$  avoiding  $U$  (which is impossible).  $\square$

For a Coxeter system  $(W, S)$  and vertex  $V \subset S$  of a visual  $M(W, S)$ -JSJ decomposition of  $(W, S)$ ,  $V$  may not be a connected subset of  $\Gamma(W, S)$ .

**Example 3.** Let  $(W, S)$  be the Coxeter system given by  $\Gamma(W, S)$  pictured in figure 3, where each edge has label 3. The only non-crossing visual virtually abelian splitting subgroups for this system are  $\langle x, y \rangle$  and  $\langle u, v \rangle$  ( $\langle x, v \rangle$  and  $\langle u, y \rangle$  are crossing). The JSJ-decomposition is given by:

$$\langle a, b, x, y \rangle *_{\langle x, y \rangle} \langle x, y, u, v \rangle *_{\langle u, v \rangle} \langle u, v, c, d \rangle$$

The set  $\{x, y, u, v\}$  generates a vertex group of this decomposition and is not connected in  $\Gamma$ .

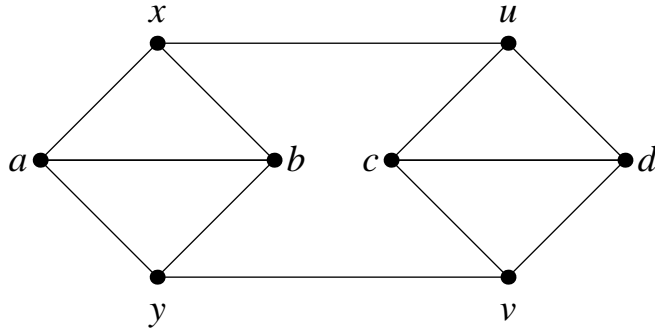


Figure 3: A disconnected JSJ vertex group

Still we have the following:

**Lemma 31** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a visual graph of groups decomposition of  $(W, S)$  that is JSJ-amenable,  $V \subset S$  is a vertex of  $\Psi$ , and  $\Psi_1$  is a visual  $M(\Psi, \langle V \rangle)$ -JSJ decomposition of  $\langle V \rangle$ . If  $D \subset V$ ,*

$\langle D \rangle \in M(\Psi, \langle V \rangle)$ ,  $D$  separates  $V$  in  $\Gamma(W, S)$ , and  $\langle D \rangle$  is non-crossing in  $M(\Psi, \langle V \rangle)$ , then there is no  $Q \subset V$  a vertex of  $\Psi_1$  such that  $D$  separates  $Q$  in  $\Gamma(W, S)$ .

Note that in the previous example, neither  $\{x, y\}$  nor  $\{u, v\}$  separates  $\{x, y, u, v\}$  in  $\Gamma$ .

**Proof:** Suppose otherwise. By lemma 30, there is  $Q' \subset V$  a vertex of  $\Psi_1$  such that  $D \subset Q'$ . As  $\Psi_1$  is  $M(\Psi, \langle V \rangle)$ -JSJ,  $D$  does not separate  $Q'$  in  $\Gamma$ . In particular,  $Q \neq Q'$ . Suppose  $F \subset V$  is such that  $\langle F \rangle$  is the group of an edge of the  $\Psi_1$ -geodesic connecting the vertices  $Q$  and  $Q'$ . As  $F$  separates  $Q$  and  $Q'$  in the tree  $\Psi_1$ , (and since  $D \subset Q'$  separates  $Q$  in  $\Gamma$ )  $F$  separates  $Q$  in  $\Gamma$ . But if  $\langle F \rangle$  is the group of the edge of this geodesic adjacent to  $Q$ , there is an induced splitting of  $\langle Q \rangle$  over  $\langle F \rangle$  compatible with  $\Psi_1$  (since  $\langle F \rangle$  is non-crossing) and  $\Psi$  (since  $\Psi$  is JSJ-amenable). This is impossible as  $\Psi_1$  is  $M(\Psi, \langle V \rangle)$ -JSJ.  $\square$

**Theorem 32** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a visual graph of groups decomposition of  $(W, S)$  that is JSJ-amenable,  $V \subset S$  is a vertex of  $\Psi$ , and  $\Psi_1$  and  $\Psi_2$  are visual  $M(\Psi, \langle V \rangle)$ -JSJ decomposition of  $\langle V \rangle$ . If  $Q \subset V$  is a vertex of  $\Psi_1$ , then  $Q$  is a vertex of  $\Psi_2$ . I.e. The vertex groups of two visual  $M(\Psi, \langle V \rangle)$ -JSJ decompositions of  $\langle V \rangle$  are the same.*

**Proof:** It suffices to show that  $Q \subset Q'$ , for  $Q'$  a vertex of  $\Psi_2$ . Otherwise, there exists  $D \subset V$  an edge of  $\Psi_2$  and  $D$  separates  $Q$  in  $\Gamma$ . This is contrary to lemma 31.  $\square$

Theorems 29 and 32 imply:

**Corollary 33** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi$  is a visual graph of groups decomposition of  $(W, S)$  that is JSJ-amenable,  $V \subset S$  a vertex of  $\Psi$ ,  $\Lambda_1$  and  $\Lambda_2$  are  $M(\Psi, \langle V \rangle)$ -JSJ decomposition of  $\langle V \rangle$ . Then there is a (unique) bijection  $\alpha$  of the vertices of  $\Lambda_1$  to the vertices of  $\Lambda_2$  such that for each vertex  $Q$  of  $\Lambda_1$ ,  $\Lambda_1(Q)$  is conjugate to  $\Lambda_2(\alpha(Q))$ .  $\square$*

**Remark 1.** For a Coxeter system  $(W, S)$ , the vertex groups of two visual  $M(W, S)$ -JSJ decompositions must be the same, but it is unreasonable to expect two such graph of groups decompositions to be exactly the same. As an example, a Coxeter diagram  $\Gamma(W, S)$  may be such that  $S = A \cup B \cup C$

where  $A \cap B = A \cap C = B \cap C = E$  and a visual  $M(W, S)$ -JSJ decomposition is given by  $\langle A \rangle *_{\langle E \rangle} \langle B \rangle *_{\langle E \rangle} \langle C \rangle$  or  $\langle A \rangle *_{\langle E \rangle} \langle C \rangle *_{\langle E \rangle} \langle B \rangle$ . See for instance example 4 of section 8.

## 7 JSJ-Decompositions

Suppose  $(W, S)$  is a finitely generated Coxeter system. Let  $\Psi_0$  be the trivial (single vertex) graph of groups decomposition of  $W$ . Then  $\Psi_0$  is vacuously JSJ-amenable. If  $\Psi_1$  is the (unique) visual  $M(W, S)(= M(\Psi_0, \langle S \rangle))$ -JSJ decomposition of  $W$  and  $E \subset S$  is such that  $\langle E \rangle$  is the group of an edge adjacent to  $V \subset S$  for  $V$  a vertex of  $\Psi_1$ , then  $\langle E \rangle \in M(\Psi_0, \langle S \rangle)$ . If  $A \subset V$  is such that  $\langle A \rangle$  is virtually abelian and  $A$  separates  $E$  in  $\Gamma(W, S)$  then lemma 22 implies  $\langle A \rangle \in M(\Psi_0, \langle S \rangle)$ . But then  $\langle E \rangle$  and  $\langle A \rangle$  are crossing in  $M(\Psi_0, \langle S \rangle)$ , contrary to the fact that  $\langle E \rangle$  is an edge group of  $\Psi_1$  (the (unique) visual  $M(W, S)(= M(\Psi_0, \langle S \rangle))$ -JSJ decomposition of  $W$ ). Instead,  $\Psi_1$  is JSJ-amenable.

If  $\langle V \rangle$  is a vertex group of  $\Psi_1$ , lemma 3 implies the (visual)  $M(\Psi_1, \langle V \rangle)$ -JSJ decomposition of  $\langle V \rangle$  is compatible with  $\Psi_1$ . Inductively assume  $\Psi_j$  is JSJ-amenable for all  $j < i$ . Let  $\Psi_i$  be obtained from  $\Psi_{i-1}$  by replacing each vertex group  $\langle V \rangle$  of  $\Psi_{i-1}$  by the (unique)  $M(\Psi_{i-1}, \langle V \rangle)$ -JSJ decomposition insured by theorem 32. By corollary 33, we may assume this decomposition is visual. The compatibility of this decomposition with  $\Psi_{i-1}$  is insured by repeated application of lemma 3. Our first result in this section will be to show  $\Psi_i$  is JSJ-amenable for all  $i$ , insuring our definitions are meaningful. We call  $\Psi_i$  the  *$i^{\text{th}}$ -stage of the JSJ-decomposition for  $(W, S)$* . As all decompositions involved are visual, there is an integer  $n$  such that for every vertex group  $\langle V \rangle$  of  $\Psi_n$ , and  $\langle D \rangle \in M(\Psi_n, \langle V \rangle)$  if  $D$  separates  $\langle V \rangle$  in  $\Gamma(W, S)$  then  $\langle D \rangle$  is crossing. In this case  $\Psi_n = \Psi_{n+1}$  and we define  $\Psi_n$  to be a *JSJ-decomposition of  $W$* . This decomposition is unique in the sense of corollary 33. In particular, if  $(W, S)$  and  $(W, S')$  are finitely generated Coxeter systems, then the JSJ-decompositions of  $W$  derived from  $(W, S)$  and  $(W, S')$  have conjugate vertex groups.

**Proposition 34** *Suppose  $(W, S)$  is a finitely generated Coxeter system. Then  $\Psi_n$ , the  $n^{\text{th}}$ -stage of the JSJ-decomposition for  $(W, S)$  is JSJ-amenable for all  $n$ .*

**Proof:** We have shown  $\Psi_0$  and  $\Psi_1$  are JSJ-amenable. Assume  $\Psi_j$  is JSJ-amenable for all  $j < n$ . Then  $\Psi_n$  is a visual graph of groups and each edge group of  $\Psi_n$  is a member of  $M(\Psi_j, \langle Q \rangle)$  for  $j < n$ . Suppose  $V \subset S$  is a vertex of  $\Psi_n$  and  $E \subset S$  is such that  $\langle E \rangle$  is the group of an edge of  $\Psi_n$  adjacent to  $V$ . We must show there is no  $A \subset V$  such that  $\langle A \rangle$  is virtually abelian and  $A$  separates  $E$  in  $\Gamma$ . Suppose otherwise. We may assume  $\langle E \rangle$  is the group of an edge of the  $M(\Psi_j, \langle U \rangle)$ -JSJ decomposition of  $\langle U \rangle$  for  $U$  a vertex of  $\Psi_j$ , where  $j < n$  and  $V \subset U$ . If  $j < n - 1$ , then  $A$  cannot separate  $E$  in  $\Gamma$  since  $\Psi_j$  is JSJ-amenable. If  $j = n - 1$ , and  $A$  separates  $E$  in  $\Gamma$ , then lemma 22 implies  $A \in M(\Psi_{n-1}, \langle U \rangle)$  and that  $\langle A \rangle$  and  $\langle E \rangle$  are crossing in  $M(\Psi_{n-1}, \langle U \rangle)$ . But then  $\langle E \rangle$  is not the group of the edge of the  $M(\Psi_{n-1}, \langle U \rangle)$  decomposition of  $\langle U \rangle$ . Instead,  $\Psi_n$  is JSJ-amenable.  $\square$

**Proposition 35** *Suppose  $(W, S)$  is a Coxeter system,  $\Psi_i$  is the  $i^{\text{th}}$ -stage of the JSJ-decomposition for  $(W, S)$ ,  $V \subset S$  is a vertex of  $\Psi_i$ ,  $\langle A \rangle \in \mathcal{C}(W, S)$  and  $B \subset V$  such that  $B$  separates  $V$  in  $\Gamma(W, S)$  and  $B \in M(\Psi_i, \langle V \rangle)$ . If  $A$  separates  $B$  in  $\Gamma(W, S)$  then  $A \subset V$ .*

**Proof:** This result is trivial for  $i = 0$  as  $V = S$  in this case. Suppose the proposition fails. Let  $i \geq 1$  be the largest integer such that for all  $j < i$ , there is no  $A \subset S$  and vertex  $V$  of  $\Psi_j$  such that  $\langle A \rangle \in \mathcal{C}(W, S)$ ,  $A$  separates  $B$  in  $\Gamma$  for some  $\langle B \rangle \in M(\Psi_j, \langle V \rangle)$ ,  $B$  separates  $V$  in  $\Gamma$ , and  $A \not\subset V$ . Then there is  $\langle A \rangle \in \mathcal{C}(W, S)$ ,  $V'$  a vertex of  $\Psi_i$ , and  $B \subset V'$  such that  $B$  separates  $V'$  in  $\Gamma$ ,  $\langle B \rangle \in M(\Psi_i, \langle V' \rangle)$ ,  $A$  separates  $B$  in  $\Gamma$ , and  $A \not\subset V'$ . Assume  $\{b_1, b_2\} \subset B$  and  $A$  separates  $b_1$  from  $b_2$  in  $\Gamma$ . By lemma 22,  $B - \{b_1, b_2\} = A - \{a_1, a_2\} \equiv M$  where  $B$  separates  $a_1$  and  $a_2$  in  $\Gamma$ . Let  $V$  be the vertex group of  $\Psi_{i-1}$  containing  $V'$ , and  $\Psi'_i$  the visual  $M(\Psi_{i-1}, \langle V \rangle)$ -JSJ decomposition that splits  $\langle V \rangle$  to give  $\Psi_i$ .

We show  $B \notin M(\Psi_{i-1}, \langle V \rangle)$ . Assume otherwise. As  $A$  separates  $B$  in  $\Gamma$  the definition of  $i$  implies that  $A \subset V$ . Lemma 22 implies  $A \in M(\Psi_{i-1}, \langle V \rangle)$ . As  $M \cup \{a_1, a_2\} = A \not\subset V'$  and  $M \subset B \subset V'$ , we assume  $a_1 \notin V'$ . Then as  $\Psi'_i$  is a tree, there is  $D \subset V$  such that  $\langle D \rangle$  is the group of an edge of  $\Psi'_i$  adjacent to  $V'$  such that any path in  $\Gamma$  from  $a_1$  to  $V'$  intersects  $D$ . Thus,  $M \subset D$ . Any path from  $a_1$  to  $a_2$  intersects  $B \subset V'$ , and so such a path intersects  $D$ . As  $D \subset V'$ ,  $a_1 \notin D$ . If  $a_2 \notin D$ , then  $D$  separates  $a_1$  and  $a_2$  in  $\Gamma$ , so that  $D$  and  $A$  are crossing in  $M(\Psi_{i-1}, \langle V \rangle)$ . But no edge of  $\Psi'_i$  is crossing in  $M(\Psi_{i-1}, \langle V \rangle)$ . Instead,  $a_2 \in D \subset V'$ . As  $\{b_1, b_2\} \subset V'$ , and since any path from  $b_1$  to  $b_2$  contains  $a_1$  or  $a_2$ , any path from  $b_1$  to  $b_2$  intersects  $D$ .

Hence, as  $\langle D \rangle$  is non-crossing in  $M(\Psi_{i-1}, \langle V \rangle)$ , and since  $B \in M(\Psi_{i-1}, \langle V \rangle)$ ,  $\{b_1, b_2\} \cap D \neq \emptyset$ . Furthermore,  $\{b_1, b_2\} \not\subset D$ , since otherwise,  $A$  would cross (the non-crossing)  $D$  in  $M(\Psi_{i-1}, \langle V \rangle)$ . We assume  $b_1 \in D$  and  $b_2 \in V' - D$ . The set  $\{a_2\} \cup M$  does not separate  $V$  in  $\Gamma$  as  $A \in M(\Psi_{i-1}, \langle V \rangle)$ . Let  $\alpha$  be a shortest path in  $\Gamma$  from  $b_1$  to  $b_2$ , avoiding  $\{a_2\} \cup M$ . Then  $\alpha$  contains  $a_1$ , and so the segment of  $\alpha$  from  $a_1$  to  $b_2$  contains  $d \in D$ . The segment of  $\alpha$  from  $d$  to  $b_2$  avoids  $A$ . Now  $\{b_1, d\} \subset D - (\{a_2\} \cup M)$ . There is no edge connecting  $b_1$  and  $d$ , since  $b_1$  cannot be connected to  $b_2$  avoiding  $A$ . As  $\langle D \rangle$  is virtually abelian, theorem 13 implies  $a_2$  is connected to  $b_1$  and to  $d$  by an edge (labeled 2). The segment of  $\alpha$  from  $a_1$  to  $d$  followed by the edge from  $d$  to  $a_2$  avoids  $B$ , which is nonsense. Instead,  $\langle B \rangle \notin M(\Psi_{i-1}, \langle V \rangle)$  as desired.

Proposition 18 implies there is  $K \subset V$  such that  $K$  separates  $V$  in  $\Gamma$  and  $\langle K \rangle$  is a maximal rank element of  $M(\Psi_{i-1}, \langle V \rangle)$  such that  $E(K) \subset E(B)$ . Then  $\langle E(K) \rangle$  has infinite index in  $B$ . As above we assume  $a_1 \notin V'$  and there is  $D \subset V$  such that  $\langle D \rangle$  is the group of an edge of  $\Psi'_i$  adjacent to  $V'$  such that any path in  $\Gamma$  from  $a_1$  to  $V'$  intersects  $D$ . In particular  $M \subset D$ .

We show  $E(K) = E(M) = E(D)$ . If  $\{b_1, b_2\} \subset K$ , then  $A$  separates  $K$  and lemma 22 implies  $Rank(K) = Rank(A) (= Rank(B))$  which is nonsense. Instead,  $E(K) = E(K \cap B) \subset E(M)$ . As  $M \subset D$ ,  $E(K) \subset E(M) \subset E(D)$ . Since it is also true that  $\{K, D\} \subset M(\Psi_{i-1}, \langle V \rangle)$ , we see that  $E(D) = E(K)$  and so  $E(K) = E(M) = E(D)$ .

Next we show  $E(M)$  is not a subset of the group of an edge of  $\Psi_{i-1}$  (adjacent to  $V$ ). Otherwise, there is a  $j < i-1$ , a vertex  $\hat{V}$  of  $\Psi_j$  containing  $V$ , and a non-crossing  $F \in M(\Psi_j, \langle \hat{V} \rangle)$  such that  $F \subset V$ , and  $E(M) \subset F$ . But then  $K$  is non-crossing in  $M(\Psi_j, \langle \hat{V} \rangle)$  and as  $K$  separates  $V$  in  $\Gamma$ , the visual  $M(\Psi_j, \langle \hat{V} \rangle)$ -JSJ decomposition of  $\langle \hat{V} \rangle$  was not maximal - which is nonsense.

As  $E(M)$  is not a subset of the group of an edge of  $\Psi_{i-1}$  adjacent to  $V$ ,  $A \subset V$ . If  $a_2 \notin V'$ , then there exists  $D' \subset V$  such that  $\langle D' \rangle$  is the group of an edge of  $\Psi'_i$ ,  $D'$  separates  $a_2$  from  $V'$  in  $\Gamma$ , and  $E(M) = E(D')$ . Note that  $D - E(M)$  and  $D' - E(M)$  generate finite groups. In particular, these sets define complete subgraphs of  $\Gamma$ . As  $\langle B \rangle \in M(\Psi_i, V')$ ,  $E(M)$  does not separate  $b_1$  from  $b_2$  in  $\Gamma$ . If  $\alpha$  is a shortest path in  $\Gamma$  from  $b_1$  to  $b_2$  avoiding  $E(M)$ , then  $a_1$  or  $a_2$  is a vertex of  $\alpha$ . If  $a_1$  is a vertex of  $\alpha$ , then some vertex  $t$  of  $\alpha$ , between  $b_1$  and  $a_1$  belongs to  $D - E(M)$  and some vertex  $s$  of  $\alpha$  between  $a_1$  and  $b_2$  belongs to  $D' - E(M)$ . But there is an edge between  $t$  and  $s$  contradicting the minimality of  $\alpha$ . Similarly if  $a_2$  is a vertex of  $\alpha$ . Instead, either  $a_1$  or  $a_2$  is an element of  $V'$ . If  $a_2 \in V'$ , then since  $B \in M(\Psi_i, \langle V' \rangle)$ ,

$\{a_2\} \cup M$  does not separate  $V'$  in  $\Gamma$ . Let  $\alpha$  be a shortest path from  $b_1$  to  $b_2$  in  $\Gamma$ , avoiding  $\{a_2\} \cup M$ . Then  $a_1$  is a vertex of  $\alpha$  and as before, there exists  $t$  and  $s$  in  $D - E(M)$  such that  $t$  is between  $b_1$  and  $a_1$  and  $s$  is between  $a_1$  and  $b_2$ . The edge joining  $t$  and  $s$  shortens  $\alpha$  which is impossible.  $\square$

**Theorem 36** *Suppose  $\Psi$  is the visual JSJ-decomposition for  $(W, S)$   $\langle V \rangle$  is a vertex group of  $\Psi$ ,  $\Phi$  is an arbitrary graph of groups decomposition of  $W$  with virtually abelian edge groups, and  $T$  is the Bass-Serre tree for  $\Phi$ . If  $\langle A \rangle \in \mathcal{C}(W, S)$ , then*

1.  *$A$  cannot separate  $B$  in  $\Gamma(W, S)$  if  $\langle B \rangle$  is an edge group of  $\Psi$ ,*
2. *the decomposition of  $\langle V \rangle$  induced by its action on  $T$  is compatible with  $\Psi$ , and*
3. *if  $M(\Psi, \langle V \rangle)$  contains no crossing elements, then  $\langle V \rangle$  stabilizes a vertex of  $\Phi$ .*

**Proof:** If 1) fails, there is an integer  $i$  and a vertex group  $\langle V \rangle$  of  $\Psi_i$ , the  $i^{\text{th}}$ -level visual JSJ-decomposition for  $(W, S)$ , such that  $\langle B \rangle$  is an edge group of  $\Psi'_{i+1}$  the  $M(\Psi_i, \langle V \rangle)$  visual JSJ-decomposition of  $\langle V \rangle$ . By proposition 35,  $A \subset V$  and so by lemma 22,  $\langle A \rangle$  and  $\langle B \rangle$  are crossing in  $M(\Psi_i, \langle V \rangle)$ . But this is impossible as no edge of  $\Psi'_{i+1}$  is crossing in  $M(\Psi_i, \langle V \rangle)$ .

For the second part of the theorem, assume  $B$  is an edge of  $\Psi$  and  $\langle B \rangle$  does not stabilize a vertex of  $T$  (equivalently,  $\langle B \rangle$  is not a subgroup of a conjugate of a vertex group of  $\Phi$ ). Then the visual decomposition of  $(W, S)$  for  $\Phi$  (given by the construction for theorem 7) has an edge group  $\langle A \rangle$  such that  $A$  separates  $B$  in  $\Gamma$  - which is impossible.

For the third part of the theorem, let  $\Lambda$  be the decomposition of  $\langle V \rangle$  determined by its action on  $\Phi$ . Since  $\Lambda$  is compatible with  $\Psi$ , if  $\Lambda$  is non-trivial, then  $M(\Psi, \langle V \rangle)$  is non-empty. But as  $\Psi$  is JSJ,  $M(\Psi, \langle V \rangle)$  can only contain crossing elements.  $\square$

## 8 Orbifold groups

If  $\Psi$  is the JSJ-decomposition of  $(W, S)$  and  $\langle V \rangle$  is a vertex group of  $\Psi$  such that  $M(\Psi, \langle V \rangle)$  contains no crossing members, then  $\langle V \rangle$  is indecomposable with respect to its action on the Bass-Serre tree for any splitting of  $W$  over virtually abelian subgroups. If instead,  $M(\Psi, \langle V \rangle)$  contains crossing members, then we say  $\langle V \rangle$  is an *orbifold vertex group* of  $\Psi$ .

**Theorem 37** *Suppose  $(W, S)$  is a finitely generated Coxeter system,  $\Psi$  is the (visual) JSJ-decomposition of  $(W, S)$ , and  $\langle V \rangle$  is an orbifold vertex group of  $\Psi$ . Then  $\langle V \rangle$  decomposes as  $\langle T \rangle \times \langle M \rangle$  where  $T \cup M = V$ ,  $M$  generates a virtually abelian group and the presentation diagram of  $T$  is either a loop of length  $\geq 4$  (in which case  $T$  generates a group that is virtually a closed surface group) or the presentation diagram of  $T$  is a disjoint union of vertices and simple paths (in which case  $T$  generates a virtually free group with graph of groups decomposition such that each vertex group is either  $\mathbb{Z}_2$  or finite dihedral and each edge group is either trivial or  $\mathbb{Z}_2$ ).*

**Proof:** As  $\Psi$  is JSJ, the only visual subgroups of  $M(\Psi, \langle V \rangle)$  are crossing. Say  $\langle A \rangle$  and  $\langle B \rangle$  are minimal rank elements of  $M(\Psi, \langle V \rangle)$  that cross one another. If  $C \subset V$  such that  $\langle C \rangle$  is virtually abelian, and  $C$  separates  $V$  in  $\Gamma$  then  $\text{Rank}(C) \geq \text{Rank}(A)$ . If additionally  $\text{Rank}(C) = \text{Rank}(A)$  then  $\langle C \rangle \in M(\Psi, \langle V \rangle)$  and so  $\langle C \rangle$  is crossing in  $M(\Psi, \langle V \rangle)$ .

By lemma 22,  $A = \{a_1, a_2\} \cup M$  and  $B = \{b_1, b_2\} \cup M$  where  $\langle M \rangle$  is virtually abelian and commutes with  $\{a_1, a_2, b_1, b_2\}$ ,  $A$  separates  $b_1$  and  $b_2$  in  $\Gamma$  and  $B$  separates  $a_1$  and  $a_2$  in  $\Gamma$ . By Proposition 23,  $\Gamma - A$  has exactly two components which intersect  $V$  non-trivially,  $C_{b_1}$  containing  $b_1$  and  $C_{b_2}$  containing  $b_2$ . Then  $V - M \subset \{a_1\} \cup C_{b_1} \cup \{a_2\} \cup C_{b_2}$ . By lemma 20 there is an edge connecting (cyclically) adjacent members of this union, and no edge connecting non-adjacent members. In particular,  $C_{b_1} \cup \{a_1, a_2\}$  is a connected subset of  $\Gamma$ . We show:

(i)  $(C_{b_1} \cup \{a_1, a_2\}) - \{b_1\}$  has exactly two components which intersect  $V$  non-trivially, one containing  $a_1$  and the other containing  $a_2$ .

A path connecting  $a_1$  and  $a_2$  in  $C_{b_1} \cup \{a_1, a_2\}$  avoiding  $b_1$  also avoids  $b_2$  and  $M$ , but this is impossible as  $B$  separates  $a_1$  and  $a_2$  in  $\Gamma$ . Hence  $b_1$  separates  $a_1$  and  $a_2$  in  $(C_{b_1} \cup \{a_1, a_2\})$ . Let  $C_{b_1, a_1}$  (respectively  $C_{b_1, a_2}$ ) be the components of  $(C_{b_1} \cup \{a_1, a_2\}) - \{b_1\}$  containing  $a_1$  (respectively  $a_2$ ). If  $z \in V$  is an element of a third component of  $(C_{b_1} \cup \{a_1, a_2\}) - \{b_1\}$ , then  $z \in \Gamma - B$  and by lemma 22, there is a (shortest) path  $\alpha$  in  $\Gamma - B$  from  $z$  to either  $a_1$  or  $a_2$ . Without loss, say  $\alpha$  connects  $z$  to  $a_1$ . Then  $a_2$  is not a point of  $\alpha$ . Note that  $\alpha$  is not contained in  $C_{b_1} \cup \{a_1, a_2\}$  since  $b_1$  separates  $z$  and  $a_1$  in  $C_{b_1} \cup \{a_1, a_2\}$  and  $\alpha$  avoids  $b_1$ . If  $\beta$  is the longest initial segment of  $\alpha$  in  $C_{b_1} \cup \{a_1, a_2\}$  with end point  $t$ , and  $s$  follows  $t$  on  $\alpha$ , then  $\beta$  avoids  $A$  and  $s \notin M \cup \{a_1, a_2\} \cup C_{b_1}$ . Then  $s$  is connected to  $z \in V$  by a path in  $\Gamma - A$ , implying  $s \in C_{b_1} \cup C_{b_2}$ . So  $s \in C_{b_2}$ . But then the edge  $[ts]$  connects points

of  $C_{b_1}$  and  $C_{b_2}$ , which is impossible (recall  $A$  separates  $C_{b_1}$  and  $C_{b_2}$  in  $\Gamma$ ). So (i) is proved.

Observe that  $V$  is contained in the union  $C_{b_1,a_1} \cup C_{b_1,a_2} \cup C_{b_2,a_1} \cup C_{b_2,a_2} \cup M \cup \{b_1, b_2\}$ . The only overlap among these sets is  $C_{b_1,a_i} \cap C_{b_2,a_i} = \{a_i\}$ . Next we show:

(ii) For  $u \in \{b_1, b_2\}$  and  $v \in \{a_1, a_2\}$ , either  $V \cap C_{u,v} = \{v\}$ , or in  $\Gamma$ ,  $\{u, v\} \cup M$  separates points of  $V \cap (C_{u,v} - \{v\})$  from points of  $V \cap (C_t \cup C_{u,s})$  where  $\{t\} = \{b_1, b_2\} - \{u\}$  and  $\{s\} = \{a_1, a_2\} - \{v\}$ .

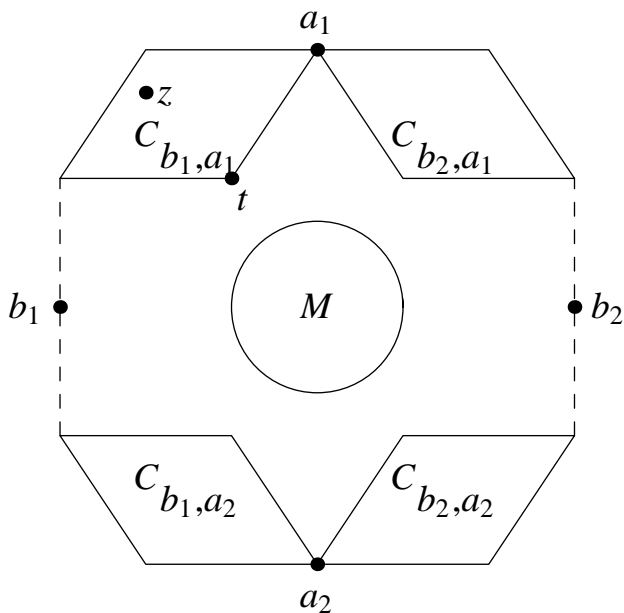


Figure 4: Circular decomposition of  $V$  in  $\Gamma(W, S)$ .

By the symmetry of the situation pictured in figure 4, it suffices to consider the case  $u = b_1$  and  $v = a_1$ . By lemma 20 there is an edge from  $a_2$  to  $C_{b_2}$  so  $C_{b_2} \cup C_{b_1,a_2}$  is a connected subset of  $\Gamma - (\{b_1, a_1\} \cup M)$ . Suppose  $z \in V \cap (C_{b_1,a_1} - \{a_1\})$  and  $\alpha$  is a path from  $z$  to  $a_2$  in  $\Gamma - (\{a_1, b_1\} \cup M)$ . Let  $t$  be the last vertex of  $\beta$ , the longest initial segment of  $\alpha$  in  $C_{b_1,a_1} - \{a_1\}$  (then  $t \neq a_2$ ) and  $s$  the vertex of  $\alpha$  following  $t$ . Note that  $s \notin \{a_1, b_1\} \cup M$  by the definition of  $\alpha$ ,  $s \notin C_{b_1,a_1} - \{a_1\}$  by the definition of  $\beta$ , and  $s$  is not an element

of a component of  $(C_{b_1} \cup \{a_1, a_2\}) - \{b_1\}$  other than  $C_{b_1, a_1}$ , since  $b_1$  separates  $C_{b_1, a_1}$  from those components in  $C_{b_1} \cup \{a_1, a_2\}$ . Hence  $s \notin C_{b_1} \cup \{a_1, a_2\} \cup M$ . Then  $s \in C_{b_2}$ , but there is no edge from  $C_{b_1}$  to  $C_{b_2}$ . So (ii) is proved.

For  $u, v$  as in (ii), if  $V \cap C_{u, v} \neq \{v\}$ , then  $\{u, v\} \cup M \equiv C$  separates  $V$  in  $\Gamma$ . There is no edge connecting  $u$  and  $v$  since otherwise,  $\text{Rank}(\langle C \rangle) < \text{Rank}\langle A \rangle$ . Hence  $\text{Rank}(\langle C \rangle) = \text{Rank}(\langle A \rangle)$ , and  $\langle C \rangle$  is crossing in  $M(\Psi, \langle V \rangle)$ . Say  $\langle C \rangle$  crosses  $\langle D \rangle$ . If  $D$  does not separate  $u$  and  $v$  in  $\Gamma$  then  $D$  must separate some other unrelated pair,  $m_1$  and  $m_2$  in  $M$ . But then  $D$  separates  $A$  in  $\Gamma$  and  $D$  separates  $B$  in  $\Gamma$ . By lemma 22,  $D - \{d_1, d_2\} = A - \{m_1, m_2\}$  and  $D - \{d'_1, d'_2\} = B - \{m_1, m_2\}$ . Hence  $\{a_1, a_2, b_1, b_2\} \subset D$ . As  $D$  is virtually abelian and the pairs  $a_1, a_2$  and  $b_1, b_2$  are unrelated pairs, there is an edge (labeled 2) in  $\Gamma$  connecting  $a_i$  to  $b_j$  for all  $i, j \in \{1, 2\}$ . This is nonsense since we have assumed  $u \in \{b_1, b_2\}$  and  $v \in \{a_1, a_2\}$  are unrelated. Instead  $D$  separates  $u$  and  $v$  in  $\Gamma$ .

If  $V \cap C_{b_1, a_1} \neq \{a_1\}$ , then proposition 23 implies  $\Gamma - (\{a_1, b_1\} \cup M)$  has exactly 2 components which intersect  $V$  non-trivially. Now,  $\{b_1, a_1\} \cup M$  separates  $C_{b_1, a_1} - \{a_1\}$  from (the connected set)  $C_{b_2} \cup C_{b_1, a_2}$ . Hence we have:

(iii) If  $V \cap C_{b_1, a_1} \neq \{a_1\}$  then  $C_{b_1, a_1} - \{a_1\}$  is contained in one of the two components of  $\Gamma - (\{a_1, b_1\} \cup M)$  which intersect  $V$  non-trivially and  $C_{b_2} \cup C_{b_1, a_2}$  is contained in the other.

Next we improve (iii):

(iv) If  $V \cap C_{b_1, a_1} \neq \{a_1\}$  then  $C_{b_1, a_1} - \{a_1\}$  is a component of  $\Gamma - (\{a_1, b_1\} \cup M)$ .

If  $K$  is a component of  $\Gamma - A$  other than  $C_{b_1}$  or  $C_{b_2}$  and  $\alpha$  is a path connecting  $K$  to  $C_{b_1, a_1} - \{a_1\} (\subset C_{b_1})$  in  $\Gamma - (\{a_1, b_1\} \cup M)$ , then  $\alpha$  contains  $a_2 \in C_{b_1, a_2}$  which is impossible by (iii). Suppose  $K$  is a component of  $C_{b_1} - \{b_1\}$  other than  $C_{b_1, a_1}$  and  $\alpha$  is a path from  $K$  to  $C_{b_1, a_1} - \{a_1\}$  avoiding  $\{a_1, b_1\} \cup M$ . If  $\alpha$  leaves  $C_{b_1}$ , then  $\alpha$  contains  $a_2$  which is impossible by (iii). But then  $\alpha$  is a path in  $C_{b_1}$  and must pass through  $b_1$ , which is also impossible and (iv) is verified.

We conclude that  $V - (A \cup B)$  is contained in the following disjoint union of sets, each of which either has trivial intersection with  $V$  or is a component of  $\Gamma - (A \cup B)$ .

$$(C_{b_1, a_1} - \{a_1\}) \cup (C_{b_1, a_2} - \{a_2\}) \cup (C_{b_2, a_2} - \{a_2\}) \cup (C_{b_2, a_1} - \{a_1\})$$

If  $(V \cap C_{b_i, a_j}) - \{a_j\} = \emptyset$ , then there may be an edge connecting  $b_i$  and  $a_j$ , but if no such set is empty, then two members of the following collection

are connected by an edge iff they are cyclically adjacent. I.e. they form a “loop” of sets in  $\Gamma$ :

$$\{a_1\}, C_{b_1, a_1} - \{a_1\}, \{b_1\}, C_{b_1, a_2} - \{a_2\}, \{a_2\}, C_{b_2, a_2} - \{a_2\}, \{b_2\}, C_{b_2, a_1} - \{a_1\}$$

By the symmetry of the above results, we could define  $C_{a_1, b_1}$ . If  $V \cap C_{a_1, b_1} \neq \{b_1\}$  then  $C_{a_1, b_1} - \{b_1\} = C_{b_1, a_1} - \{a_1\}$  and is the only component of  $\Gamma - (A \cup B)$  containing points of  $V$  and connected to each of  $a_1$  and  $b_1$  by an edge. Similarly for the other such sets. Remove  $C_{u, v} - \{v\}$  from the above loop if  $V \cap C_{u, v} = \{v\}$  and then sets are connected by an edge only if they are cyclically adjacent.

If  $\{u, v\} \in \{\{a_1, b_1\}, \{b_1, a_2\}, \{a_2, b_2\}, \{b_2, a_1\}\}$  and  $V \cap C_{u, v} \neq \{v\}$ , then  $\langle \{u, v\} \cup M \rangle$  crosses say  $\langle \{d_1, d_2\} \cup M \rangle$  in  $M(\Psi, \langle V \rangle)$ . By (iii) and proposition 23, for some  $i$ ,  $d_i \in C_{u, v} - \{v\}$ . By the above analysis,  $d_i$  separates  $u$  and  $v$  in  $C_{u, v} \cup \{u\}$ . Form the sets  $C_{d_i, u}$  and  $C_{d_i, v}$  as before. Note that  $d_i$  commutes with  $M$ . Continue until all remaining  $C_{u, v}$  are such that  $V \cap C_{u, v} = \{v\}$ . The the final collection of sets are all of the singletons of  $V - M$  and each commutes with  $M$ . Then  $T(\equiv V - M)$  and  $M$  satisfies the conclusion of our theorem.

If the presentation diagram for  $T$  is a loop of length  $\geq 4$ , then theorem 7.16.2 of Beardon’s book [1] guarantees the existence of an  $n$ -sided hyperbolic polygon whose vertex angles are  $\frac{\pi}{m_i}$  (in cyclic order) for  $m_i$  the edge labels of the presentation diagram (in cyclic order). Theorem 7.1.4 of [12] concludes that the reflection group in this Coxeter polygon is a Coxeter group with cyclic presentation diagram and edge labels  $m_i$  (in cyclic order). Selberg’s lemma implies this Coxeter group has a torsion free subgroup of finite index and so  $\langle T \rangle$  has a closed surface subgroup of finite index.

If  $T$  is not a loop, a visual decomposition of  $\langle T \rangle$  produces the desired graph of groups decomposition of  $\langle T \rangle$ .  $\square$

Note that for a Coxeter system  $(W, S)$   $A \subset S$  may separate  $\Gamma(W, S)$  and generate a visual virtually abelian group, but  $A$  may not be a subset of a vertex of a JSJ-decomposition with virtually abelian edge groups for  $W$ .

**Example 4.** Let  $(W, S)$  be the Coxeter system with presentation diagram shown in figure 5 (with all edge labels equal to 2). Then  $W$  has virtually abelian JSJ-decomposition:

$$\langle a, b, c \rangle *_{\langle a, c \rangle} \langle a, c, d \rangle_{\langle a, c \rangle} \langle a, c, e \rangle_{\langle a, c \rangle} \langle a, c, f \rangle$$

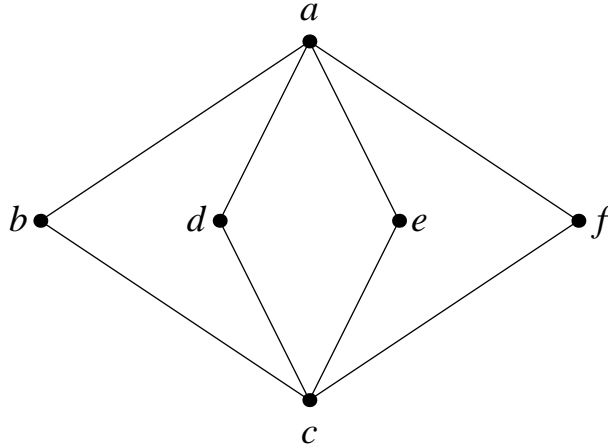


Figure 5:  $\Gamma(W, S)$

The set  $\{a, b, c, d\}$  separates  $\Gamma$ , and generates a *rank-2* virtually abelian subgroup of  $W$ . Each vertex group of the JSJ-decomposition is *rank-1* and so cannot contain a conjugate of  $\langle a, b, c, d \rangle$ .

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