Categories of Partial Frames

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Abstract. This article discusses the basic categorical algebra for categories of partial frames. Categories of partial frames are labelled by subset selectors that indicate which joins exist. Constructions for limits, colimits, and free functors connecting various categories of partial frames are given.

Examples of partial frame categories are given. Subset selectors which preserve surjections are virtually the same as rules which select all subsets smaller than a given cardinal.

Preface

The article's goal is to describe categories of partial frames. A partial frame is a meetsemilattice in which certain distinguished joins exist and finite meets distribute over distinguished joins. This is made precise in Subsection 2.1.

Particular types of "partial frames" have already appeared in frame theoretic literature. Madden [19] and Madden & Molitor [20] use κ-frames for any regular cardinal κ to draw useful frame-theoretic conclusions: many monoreflections on the category of Tychonoff locales are produced, and epimorphisms of frames (monomorphisms of locales) are identified. Johnstone and Vickers [11] and Banaschewski [1] and [4] use preframes – meet semilattices with directed joins and a distributive law – to simplify calculation of colimits of frames and produce "the shortest known proof of the Tychonoff Theorem."

The article of Paseka [24] has similar goals to this article, so a key difference deserves mention. Paseka only considers subset selectors $\mathcal{F}$ which have the feature that if $f : A \to B$ is a surjective meetsemilattice homomorphism, $\mathcal{F}B = \{f(S) : S \in \mathcal{F}A\}$. Paseka gives no examples other than cardinalities which satisfy this condition, and Propositions 37 and 39 (proved below) make me doubt that there are other examples. If this is the case, [24] does not prove anything that was not already in [19].

In contrast, this article's definition of subset selector (Definition 26) is more general. The last section of the paper describes several interesting classes of subset selectors. Moreover, the machinery developed in this article does not depend on the precise definition of meetsemilattice. For example it does not depend on whether one requires meetsemilattices to have a top, bottom or any combination thereof. In fact, most of the important results still hold if one replaces meetsemilattices with

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κ-complete meetsemilattices. Allowing these variations in the definition of meetsemilattice gives several more classes of examples. See Section 3 for our discussion of examples of subset selectors.

Section 1 develops a basic theory of subfunctors, reviews monad theory, proves that any subfunctor of a monad can be extended to a submonad, and reviews the basic facts about the category $\mathcal{M}$ of meetsemilattices. Section 2 describes the free frame monad $\mathcal{D}$ over $\mathcal{M}$. A partial frame is a Z-algebra, where Z is any submonad of $\mathcal{D}$. Section 2 also describes image factorization, limits and colimits in the category $Z\mathbf{Frm}$ of Z-frames. The crucial result, Theorem 32, which allows computation of colimits is generalized from Meseguer [22] and Banaschewski and Nelson [3]. This theorem also shows that for $Z_1 \subseteq Z_2$ there is a free functor $Z_1\mathbf{Frm} \to Z_2\mathbf{Frm}$ whose unit is injective; every Z_1-frame can be embedded in a Z_2-frame.

Applications of partial frames to topology are not considered in this article.

This article refines and refocuses some results of the author's doctoral dissertation [28]. The electronic version of the dissertation will remain freely available.

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1. Preliminaries

This section develops some category theoretic and algebra results, convenient for discussing subset selectors. Subsection 1.1 contains a discussion of subfunctors in a extremally wellpowered (epi, extremal mono)-category; the class of subfunctors of a functor behaves very much like the power set of a set. In fact, arbitrary classes of subfunctors have meets and joins. Subsection 1.2 reviews the basic facts about monads and their algebras. Subsection 1.3 shows that if T is a monad, the class of submonads is a closure family on the class of subfunctors of T. A criterion is given for when an algebra over a subfunctor extends to an algebra over its monadic closure. Subsection 1.4 reviews basic facts about meetsemilattices. The treatment of subfunctors in Subsection 1.1 and the discussion of monadic closure on the lattice of subfunctors in Subsection 1.3 are new.

1.1. Subfunctors. The reader is assumed familiar with natural transformations and factorization systems; background on these subjects can be found in [8], [18] and [10].

1. This article makes use of both vertical composition of natural transformations (which takes natural transformations $\alpha : F \to G$ and $\beta : G \to H$ as input and outputs $\beta \cdot \alpha : F \to H$) and horizontal composition (which takes natural transformations $\alpha : F \to G$ and $\beta : H \to I$ as input and outputs $\beta\alpha : HF \to IG$). We freely use pasting conventions: i.e., F may denote both a functor and the identity natural transformation $F \to F$, the interchange law shows that if two squares of natural

transformations commute their composition also commutes. For background see Barr and Well [6]. In writing equations with both compositions, we give horizontal composition a higher precedence so that (for example) $\alpha\alpha \cdot \beta$ means $(\alpha\alpha) \cdot \beta$.

Define an *SF-category* – short for, category where subfunctors behave well – to be a complete, extremally wellpowered, (epi, extremal mono)-category.

Lemma 2. [10] *Any SF-category $\mathcal{A}$ has the following features.*

(i) *Each map f factors as $f = m \cdot e$ with e epi and m extremal mono; the factorization is unique in the sense that if $m'e'$ and me are two such, there is an isomorphism i with $i \cdot e = e'$ and $m' \cdot i = m$.*
(ii) *If $a : A \to B$ and $b : B \to C$ are extremal mono, then so is $b \cdot a$.*
(iii) *If $m = a \cdot b$ is extremal mono, then b is also extremal mono.*
(iv) *Extremal monos are pullback stable.*
(v) *Intersections of extremal monos are extremal mono.*
(vi) *(diagonalization) If $g \cdot e = m \cdot f$, e is epi, and m is extremal mono, then there is k with $m \cdot k = g$ and $k \cdot e = f$.*

3. The phrases "$A \to B$ is extremal mono" and "A is an extremal subobject of B" mean the same thing, by definition. (We have a similar convention with monomorphisms and subobjects.) Define $\mathrm{Sub}(A)$ to be the class of extremal subobjects with codomain A. $\mathrm{Sub}(A)$ may be preordered by saying $s_1 \subseteq s_2$ exactly when there exists k with $s_1 = s_2 k$; k is uniquely determined and extremal mono. We identify $s_1, s_2 \in \mathrm{Sub}(A)$ if $s_1 \subseteq s_2$ and $s_2 \subseteq s_1$; the k connecting extremal monos thus identified must be an isomorphism. We deliberately de-emphasize the distinction between particular extremal monomorphisms into A and their equivalence classes. $\mathrm{Sub}(A)$ is a complete lattice, with operations $\cup$ – join – and $\cap$ – meet. See [8, volume 1, 4.2].

4. Any map $f : A_1 \to A_2$ induces an adjoint connection between $\mathrm{Sub}(A_1)$ and $\mathrm{Sub}(A_2)$. The map $f^{+1} : \mathrm{Sub}(A_1) \to \mathrm{Sub}(A_2)$ takes $s : S \to A_1$ to $f^{+1}s : f^{+1}(S) \to A_2$ – the second factor in the (epi, extremal mono)-factorization of $f \cdot s$. The right adjoint $f^{-1} : \mathrm{Sub}(A_2) \to \mathrm{Sub}(A_1)$ is defined by pullback. [8, volume 1, 4.2]

Lemma 5. *Suppose $f : A \to B$ is a map in an SF-category and $s : S \to A$, $t : T \to B$ are extremal subobjects. f induces a map $f|_S : S \to T$ such that $f \cdot s = t \cdot f|_S$ if and only if $f^{+1}(S) \subseteq T$. There is at most one such map $f|_S$.*

Proof. If $f|_S$ exists, one applies the diagonialization property to

$$S \to f^{+1}S \subseteq B = t \cdot f|_S$$

to conlude $f^{+1}S \subseteq T$. For the converse: if $f^{+1}S \subseteq T$, then one defines $f|_S = S \to f^{+1}S \subseteq T$. □

6. A *subfunctor* $e : E \to F$ is a natural transformation with each eA extremal mono. (Notation: the component of a natural transformation $e : E \to F$ at A is $eA : EA \to FA$.) Note that an extremal subobject assignment $A \mapsto EA \subseteq FA$ induces a subfunctor if and only if for each map $f : A \to B$, $(Ff)^{+1}(EA) \subseteq EB$.

Any subfunctor obviously gives such a map. Conversely, if e is such an extremal subobject assignment, the restriction of Ff to EA followed by $(Ff)^{+1}(EA) \subseteq EB$ serves to define Ef.

7. Given a functor F, let $\mathrm{Subfun}(F)$ denote the class of its subfunctors. We order (equivalence classes of) subfunctors by $E_1 \subseteq E_2$ if and only if for each object A, $E_1A \subseteq E_2A$. Note that $E_1 \subseteq E_2$ if and only if there is a natural transformation $k : E_1 \to E_2$ with $e_1 = e_2 \cdot k$.

Proposition 8. *Suppose $\mathcal{A}$ is an SF-category.*

1. *For any functor F with codomain $\mathcal{A}$,* $\mathrm{Subfun}(F)$ *is a complete lattice, with "pointwise" operations. In fact, each subclass of* $\mathrm{Subfun}(F)$ *has a meet and join.*
2. *If F, G have codomain $\mathcal{A}$ and $\alpha : F \to G$ is natural, then α induces an adjoint connection between* $\mathrm{Subfun}(F)$ *and* $\mathrm{Subfun}(G)$*. The adjoint connection is also defined "pointwise".*
3. *Given F and η as above, and $e : E \to F \in$* $\mathrm{Subfun}(F)$*, the map assignment z defined by (epi, extremal mono)-factorization of $\eta A \cdot eA$, $\eta A \cdot eA = (\eta A)^{+1}(eA) \cdot zA$ is natural.*
4. *If $\alpha : E \to F$ and $\beta : G \to H$ are subfunctors, and either E or F preserves extremal monomorphisms, then $\beta\alpha$ is a subfunctor.*

Proof. For (1), let $(E_i)_{i \in I}$ be an indexed class of subfunctors. Since $\mathcal{A}$ is extremally wellpowered, each family $\{E_iA\}_{i \in I}$ is, in fact, a set. Thus, we may define functions S, I assigning extremal subobjects of FA by $JA = \cup_i E_iA$ (the join of (E_i)) and $MA = \cap E_iA$ (the meet of (E_i)). Consider any map $f : A \to B$. As noted in Paragraph 4, taking images under Ff preserves joins so

$$(Ff)^{+1}(JA) = \cup_i (Ff)^{+1}(E_iA) \subseteq \cup_i (E_iB) = JB$$

and

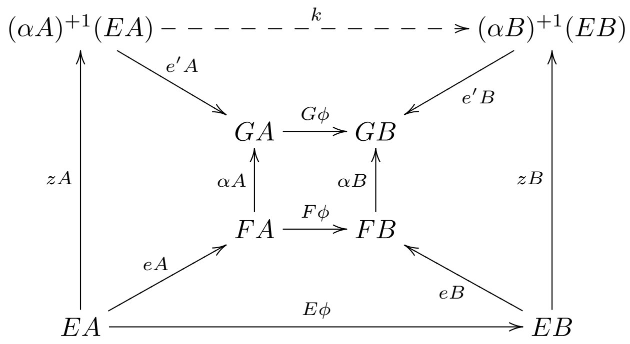
$$(Ff)^{+1}(MA) \subseteq \cap_i (Ff)^{+1}(E_iA) \subseteq \cap_i (E_iB) = MB.$$

Thus, by Paragraph 6, the extremal subobject assignments $A \mapsto JA \subseteq FA$ and $A \mapsto MA \subseteq FA$ are subfunctors.

For (2): Since any map $f : A \to B$ induces an adjoint connection between $\mathrm{Sub}(A)$ and $\mathrm{Sub}(B)$ and the order on subfunctors is pointwise (see 4 and 6) it suffices to show that

- $A \mapsto (\alpha A)^{+1}(eA)$ is a subfunctor of G whenever e is a subfunctor of F, and
- $A \mapsto (\alpha A)^{-1}(eA)$ is a subfunctor of F whenever e is a subfunctor of G.

For the first, let $\phi : A \to B$ be any map and consider the following diagram



The left and right trapezoids are obtained by factoring $\alpha B \cdot eB$ and $\alpha A \cdot eA$, respectively; in each case z is the epi part and e' is the extremal mono part. The square expresses the naturality of α. The bottom trapezoid expresses that $e : E \to F$ is a subfunctor.

It now suffices to show there is a map

$$k : (\alpha A)^{+1}(EA) \to (\alpha B)^{+1}(EB)$$

that makes the top trapezoid commute. For this, use the diagonalization property. Define $\mathbf{f} = zB \cdot E\phi$, $\mathbf{g} = G\phi \cdot e'A$, $\mathbf{e} = zA$ and $\mathbf{m} = e'B$; note that $\mathbf{g} \cdot \mathbf{e} = \mathbf{m} \cdot \mathbf{f}$ with $\mathbf{e}$ epi and $\mathbf{m}$ extremal mono. By the diagonalization property, the desired k exists.

The second bulleted assertion is proved similarly, with the existence of the missing map $(\alpha A)^{-1}(EA) \to (\alpha B)^{-1}(EB)$ obtained from the definition of a pullback.

The proof of (3) is reasonably clear. To prove (4) one uses the definition of horizontal composition of natural transformations, the fact that E or F preserves extremal monos, and that extremal monos compose. □

1.2. Monads. We briefly review the basic results on monads. For a more complete introduction, the reader would be well served by reading appropriate chapters in : MacLane [18], Manes [21], Barr and Wells [6], or Borceux [8]. There are no new results in this subsection; any unattributed result stated in this subsection may be found in the sources listed above.

9. A *monad* $\mathbf{T} = (T, \eta, \mu)$ on $\mathcal{A}$ consists of a functor

$$T : \mathcal{A} \to \mathcal{A},$$

a natural transformation

$$\eta : \mathrm{id}_{\mathcal{A}} \to T,$$

and a natural transformation

$$\mu : T^2 \to T,$$

such that the following identities (expressed by commutative diagrams) hold: the unit laws –

$$\begin{array}{ccccc} T & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & T \\ & \searrow{=} & \downarrow{\scriptstyle \mu} & \swarrow{=} & \\ & & T & & \end{array}$$

and the associative law –

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array} .$$

We will relax the notation by always refering to a monad by the name of its functor part, and always using η and μ (with appropriate superscripts, when necessary) to denote the unit and multiplication.

10. Let T be a monad on $\mathcal{A}$. A T-*algebra* (A, a) consists of $A \in \mathrm{Obj}(\mathcal{A})$ and $a : TA \to A$ (the so-called structure map) such that $a \cdot \eta A = \mathrm{id}_A$ (unit law) and

$$\begin{array}{ccc} T^2A & \xrightarrow{Ta} & TA \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow{a} & A \end{array}$$

(associative law) commutes. A homomorphism $f : (A, a) \to (B, b)$ of T-algebras is an $\mathcal{A}$-map f such that

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow{f} & B \end{array}$$

commutes. The category of all T-algebras and T-algebra homomorphisms is denoted $\mathcal{A}^T$.

11. There is a forgetful functor $G^T : \mathcal{A}^T \to \mathcal{A}$; it is defined by $G^T(A, a) = A$ and $G^T(f) = f$. The forgetful functor G^T has a left adjoint $F^T : \mathcal{A} \to \mathcal{A}^T$ defined by

$$F^T(A) = (TA, \mu A)$$

and

$$F^T(f) = Tf.$$

The associated natural transformations are

$$\eta^T = \eta : \mathrm{id}_{\mathcal{A}} \to G^T F^T = T$$

and

$$\epsilon^T : F^T G^T \to \mathrm{id}_{\mathcal{A}^T} : \epsilon^T(A, a) := a.$$

Each adjunction $(F, G, \eta : \mathcal{A} \to GF, \epsilon : FG \to \mathcal{A})$ induces a monad $(GF, \eta, G\epsilon F)$. The map

$$(T, \eta, \mu) \mapsto (F^T, G^T, \eta, \epsilon) \mapsto (G^T F^T, \eta^T, G\epsilon^T F)$$

is the identity on T, but the other composition need not be. In other words, each monad is induced by the adjunction $(F^T, G^T, \eta, \epsilon^T)$, but not every adjunction is obtained this way. An adjunction equivalent to $(F^T, G^T, \eta^T, \epsilon^T)$ for some monad T is said to be *monadic.*

Categories of algebras over monads are well-endowed with limits and colimits.

Proposition 12. [6, 3.3.4] *The functor G^T creates limits, i.e., if $D : I \to \mathcal{A}^T$ is a diagram and $G^T D$ has a limit $(L, \ell_i : L \to D(i))$ in $\mathcal{A}$, there is a unique T-structure on L making each ℓ_i a T-algebra homomorphism.*

Theorem 13. [17] *If $\mathcal{A}$ is cocomplete and $\mathcal{A}^T$ has coequalizers, then $\mathcal{A}^T$ is cocomplete.*

Definition 14. A *monad map* is a natural transformation $\alpha : S \to T$ between monads S and T, such that $\eta^T = \alpha \cdot \eta^S$ and $\alpha \cdot \mu^S = \mu^T \cdot \alpha\alpha$.

The following theorem insures that there are adjunctions between categories of monad algebras.

Theorem 15. [8, vol 2, 4.5.7, 4.5.9] *Suppose T and S are monads over $\mathcal{A}$, $\lambda : S \to T$ is a monad map, and $\mathcal{A}^T$ has coequalizers. Then there is a forgetful functor $Q : \mathcal{A}^T \to \mathcal{A}^S$ which has a left adjoint and satisfies $G^S Q = G^T$.*

1.3. From subfunctor to submonad. We define *submonad* to mean a subfunctor which is also a monad map. Theorems 16 and 21 were modeled on Meseguer [22, Prop 3.4], but formulated and proved by the author.

Theorem 16. *Let (T, η, μ) be a monad over an SF-category $\mathcal{A}$. Each subfunctor of T has a "monadic closure"; i.e., for each subfunctor F, there is a minimum subfunctor $\overline{f} : \overline{F} \to T$ with $F \subseteq \overline{F}$ and $(\overline{F}, n, m)$ a monad, where n and m are natural transformations uniquely determined by $\overline{f} \cdot m = \mu \cdot \overline{f}^2$ and $f \cdot n = \eta$.*

Proof. Let $\mathfrak{F}$ denote the class of (functor parts of) submonads of T which contain F; $\mathcal{F}$ is nonempty because it contains T. By Proposition 8, $\overline{F} = \cap\mathfrak{F}$ is a subfunctor, the meet of $\mathfrak{F}$ in the lattice of subfunctors.

If $F_0 \in \mathfrak{F}$, then by Lemma 5 and the fact that F_0 is a monad, $\mu^{+1}F_0^2 \subseteq F_0$. Since, μ^{+1} is order preserving,

$$\mu^{+1}\overline{F}^2 = \mu^{+1}(\cap\mathfrak{F}^2) \subseteq \cap\{\mu^{+1}F^2 i : F \in \mathfrak{F}\} \subseteq \cap\mathfrak{F} = \overline{F}$$

By Lemma 5, we get a unique $mA : \overline{F}^2 A \to \overline{F}A$ with $\overline{f} \cdot m = \mu \cdot \overline{f}^2$. Since μ, $\overline{f} : \overline{F} \to T$ and $\overline{f}^2 : \overline{F}^2 \to T$ are natural and $\overline{f}$ is mono, m is natural.

For $f_0 : F_0 \to T$, with $F_0 \in \mathfrak{F}$, η factors through f_0; by properties of limits, η factors through $\overline{f}$.

Uniqueness of m and n follows because $\overline{f}$ is mono. □

17. One can alternately describe $(\overline{F}, n, m)$ by transfinite induction. In this case, one defines

$$F_0 = F \cup \eta^{+1}\mathrm{id}_{\mathcal{A}},$$
$$F_{\lambda+1} = \mu^{+1}F_\lambda^2 \cup F_\lambda,$$

and when $\kappa = \vee K$ is a limit ordinal

$$F_\kappa = \cup_{\lambda\in K} F_\lambda.$$

At each stage one defines m_λ and n_λ so that $f_{\lambda+1} \cdot m_\lambda = \mu \cdot f_\lambda^2$ and $f_\lambda \cdot n_\lambda = \eta$. One also may define a natural transformation $c_\lambda : F_\lambda \to F_{\lambda+1}$. Each sequence terminates, because – for any given A – $\{E_\lambda A : \lambda \in ORD\}$ is a set, because in an SF-category each object only has a set of extremal subobjects. (ORD denotes the class of ordinals.)

18. Also note that the iterative construction described in Paragraph 17 resembles a constructions of Barr [5]. In Barr's construction, one begins with a functor $F : \mathcal{A} \to \mathcal{A}$, and generates a free monad F_{free} from F. The freeness means that there is a natural transformation $f : F \to F_{free}$ and if (S, η^S, μ^S) is any monad and $\alpha : F \to S$, then there is a unique monad map $k : F_{free} \to T$ making $\alpha = k \cdot f$.

Kůrkov-Pohlov and Koubek [15] examine a similar situation. Let $F, G : \mathrm{Set} \to \mathrm{Set}$. The article [15] considers the category whose objects are functions $FA \to GA$, and maps are functions $f : A_1 \to A_2$ making the square $a_2 \cdot Ff = Gf \cdot a_1$ commute. The article contains a necessary and sufficent condition for this category to be monadic; G must be representable and F not excessive (roughly, F does not increase powers of arbitrarily big sets). One recovers Barr's construction letting $G = \mathrm{id}_{\mathbf{Set}}$.

In some cases, Theorem 16 can be proved using Barr's results; given a subfunctor F of T make the free functor $\overline{F}$, then factor the induced monad map $\overline{F} \to T$ to produce a submonad of T. Unfortunately, this reduction does not always work, because not every functor freely generates a monad.

The monad $\mathcal{D}$ on $\mathcal{M}$ (see Paragraph 24) has this feature; the cardinality $|\mathcal{D}A|$ is at least 2^w where w is any cardinal of an antichain in A. There are posets with $w = |A|$; for such a poset the sequence of cardinalities $|\mathcal{D}^\alpha A|$ is $|A|, 2^{|A|}, 2^{2^{|A|}}, \cdots$, so the construction of Barr does not yield a free monad generated by the functor

$\mathcal{D}$. I find it very plausible that subfunctors of $\mathcal{D}$ may behave like $\mathcal{D}$ itself, in that they do not freely generate monads on $\mathcal{M}$. For these functors, the free monad construction would not apply.

Corollary 19. *Let $\mathcal{A}$ be an SF-category and T a monad on $\mathcal{A}$. The class* $\mathrm{Submon}(T)$ *of submonads of T forms a complete lattice: meets in* $\mathrm{Subfun}(T)$ *and* $\mathrm{Submon}(T)$ *agree; joins in* $\mathrm{Submon}(T)$ *are subfunctor joins followed by monadic closure. Natural transformations exhibiting inequalities are automatically monad maps.*

20. To accurately state a result of Zenk [28, Lemma 4.2.6], regarding when algebra structures extend, we develop some terminology. Let $F : \mathcal{A} \to \mathcal{A}$ be a functor and $n : \mathcal{A} \to F$ a natural transformation. An *F-algebra* is a map $a : FA \to A$ obeying a unit law $a \cdot \eta A = \mathrm{id}_A$. We say *unions of chains are colimits*, if the join of any chain of extremal subobjects is the colimit of the chain of extremal subobjects.

Theorem 21. *Suppose $\mathcal{A}$ is an SF-category and $\overline{F}$ is defined as above. Suppose $\mathcal{A}$ is cocomplete and unions of chains are colimits. Suppose (A, a) is an F-algebra. Then $a = a_0$ extends to an $\overline{F}$-algebra structure map $\overline{a} : \overline{F}A \to A$ if and only if for each ordinal λ there is a unique $a_{\lambda+1}$ such that*

$$\begin{array}{ccc} F_\lambda^2(A) & \xrightarrow{F_\lambda a_\lambda} & F_\lambda(A) \\ \downarrow{\scriptstyle m_\lambda} & & \downarrow{\scriptstyle a_\lambda} \\ F_{\lambda+1}(A) & \xrightarrow{a_{\lambda+1}} & A \end{array}$$

commutes.

1.4. Meetsemilattices.

A *meetsemilattice* is a poset in which each finite subset has a meet. Alternatively, a meetsemilattice could be defined to be a set with an idempotent, commutative, and associative operation $\wedge$, with an identity 1; in this view the order is defined by $x \le y$ iff $x \wedge y = x$. Homomorphisms are maps which preserve finite meets. In particular, a meetsemilattice has a top element 1, which is preserved by homomorphisms.

Lemma 22. *In $\mathcal{M}$*

1. *$f : A \to B$ is epi if and only if f is onto;*
2. *f is extremal mono if and only if f is mono if and only if f is injective;*
3. *Surjective $\mathcal{M}$-homomorphisms f with domain A are in bijective correspondence with $\mathcal{M}$-congruences on A.*

Thus, $\mathcal{M}$ is an SF-category.

Proof. (Sketch) Throughout, 2 denotes the two-element meetsemilattice $\{0, 1\}$ with $0 \wedge 1 = 0$.

(1) Epimorphisms are onto, because if $f : A \to B$ isn't onto – say $b \in B \setminus f(A)$, then the characteristic functions of $\uparrow b$ and $\uparrow (\uparrow b \cap f(A))$ are meetsemilattice homomorphisms $g, h : B \to 2$ with $gf \neq hf$.

(2) To show monomorphisms are injective, we give a bijection between the elements of a meetsemilattice A and maps $2 \to A$. Any map $2 \to A$ sends 1_2 to 1_A. For each $a \in A$, we define $x : 2 \to A$ by $x(0) = a$ and $x(1) = 1$; this is obviously a bijection. Each monomorphism is extremal because if f is mono and $f = a \cdot b$ with b epi, then b is a bijective meetsemilattice homomorphism. Thus, b must be an isomorphism.

(3) is obvious. □

23. By Proposition 12, using that $\mathcal{M}$ is monadic over **Set**, limits in $\mathcal{M}$ are computed as in **Set**. Colimits are fairly simple to compute: the coequalizer of $f, g : A \to B$ is the quotient by the smallest congruence which identifies all pairs $(f(a), g(a))$ for $a \in A$. The coproduct of a family $(A_i : i \in I)$ of meetsemilattices consists of all finite formal meets, where each factor comes from some A_i; coproduct injections $A_i \to \oplus A_i$ send $x \in A_i$ to the obvious 1-element formal meet.

2. Partial Frames

2.1. Subset Selectors.

When dealing with a poset A, a subset $S \subseteq A$ and $x \in A$ we use the notation

$$\downarrow S := \{x \in A : \exists s \in S,\, x \leq s\}$$

and

$$\downarrow x := \downarrow \{x\};$$

we refer to $\downarrow S$ as the *downward closure* of S.

24. The *free frame monad* $\mathcal{D}$ on $\mathcal{M}$ consists of: the functor $\mathcal{D}$ which sends each meetsemilattice to

$$\mathcal{D}A = \{\downarrow S : S \subseteq A\}$$

and acts on maps by taking downward closures of images; the unit $\eta^{\mathcal{D}}$ defined by

$$\eta^{\mathcal{D}} A(x) = \downarrow x;$$

and, the multiplication $\mu^{\mathcal{D}}$ defined by

$$\mu^{\mathcal{D}} A(\mathfrak{S}) = \cup \mathfrak{S}.$$

The following fact is obvious, but useful.

Lemma 25. *Let A be a poset and $B \subseteq \mathcal{D}A$ with*

$$\forall x \in A,\, \downarrow x \in B.$$

If $f : B \to A$ is an order preserving map such that for each $x \in A$ $f(\downarrow x) = x$, then for each $S \in B$, $f(S) = \vee S$.

Definition 26. A *subset selector* is a nontrivial submonad $Z \subseteq \mathcal{D}$. Explicitly, a subset selector Z is a rule which assigns each $\mathcal{M}$-object a family ZA of down sets, such that:

(1) For each meetsemilattice A and each $x \in A$, $\downarrow x \in A$.
(2) If $S, T \in ZA$ then $S \cap T \in ZA$.
(3) If $f : A \to B$ is any $\mathcal{M}$-map

$$\{\downarrow f(S) : S \in ZA\} \subseteq ZB.$$

(4) If $\mathfrak{S} \in ZZA$, then $\cup\mathfrak{S} \in ZA$. (This condition ensures that $\mu^{\mathcal{D}}$ induces a multiplication μ^{Z}.)

If Z is a subset system, and $S \in ZA$ we say Z *selects* S. This shorthand language is slightly ambigious because, for example, for some subset selector Z and meetsemilattice A it is concievable that there are subsets S and S' which are isomorphic as posets and $S \in ZA$ but $S' \notin ZA$. The examples discussed in this article do not have this sort of pathology, and in this article, little confusion should arise from expressions like "Z selects all finite sets", etc.

The class of subset selectors forms a complete lattice; in fact, meets and joins of arbitrary classes of subset selectors exist by Proposition 8.

For any subset selector Z let Z**Frm** denote the category $\mathcal{M}^{Z}$ of Z-algebras. We refer to the objects in Z**Frm** as *Z-frames.* A meetsemilattice map $f : A \to B$, where A is a Z-frame, is *Z-complete* if for each $S \in ZA$, $f(\vee S) = \vee f(S)$.

By virtue of the fact that we consider Z as a monad over $\mathcal{M}$, some meet/join distributivity occurs. Specifically, the structure map $a : ZA \to A$ (which is defined by $a(S) = \vee S$) is a meetsemilattice ' homomorphism. Thus, $\vee(S \cap T) = \vee\{s \wedge t : s \in S, t \in T\} = (\vee S) \wedge (\vee T)$. The article Beck [7] discusses monad distributive laws, and shows that monad distributive laws are equivalent to liftings (i.e., situations where S and T are monads over $\mathcal{A}$ and T induces a monad on $\mathcal{A}^{S}$). The monad $\mathcal{D}$ is a lifting of a monad over the category of posets.

The following proposition should be clear from the discussion above.

Proposition 27. *Z**Frm** has free objects. Z**Frm** consists of all meetsemilattices in which each Z-set has a join and finite meets distribute over joins, and all Z-complete maps between them.*

28. Recall that a *closure operator* on a poset A is a map $n : A \to A$ satisfying $a \leq b \implies n(a) \leq n(b)$, $a \leq n(a)$ and $n(n(a)) = n(a)$. If A is complete, there is a bijection between closure operators on A and subsets of A closed under arbitrary meets : $S \mapsto n_S : n_S(x) = \wedge\{s \in S : x \leq s\}$.

29. Recall a subposet $A \leq B$ is *join dense* if for each $b \in B$,

$$b = \vee(\downarrow b \cap A);$$

in this case, we say B is a *join dense extension of* A. Let $A \leq B$ be a join dense extension, where B is complete. By join density, B is order isomorpic to the set

$$\mathcal{D}_B A = \{\downarrow b \cap A : b \in B\}$$

ordered by inclusion.

Since B is complete and

$$\downarrow \wedge S = \cap \downarrow S,$$

there is a closure operator $n_B : \mathcal{D}A \to \mathcal{D}A$ such that $\mathcal{D}_B A = n_B \mathcal{D}A$. Thus, each complete join dense extension is isomorphic (over A) to an extension of the form

$$A \to n\mathcal{D}A : x \mapsto \downarrow x,$$

for some closure operator n on $\mathcal{D}A$. (Note that in any join dense extension, the sets associated with $x \neq y \in A$ are distinct, so $n(\downarrow x) = \downarrow x$ for any $x \in A$.)

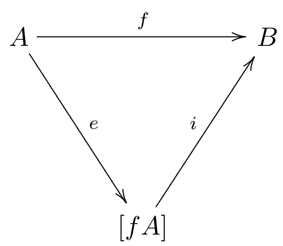
Similarly, each map $f : A \to B$, where B is complete and $f(A)$ is join dense in B is isomorphic to

$$A \mapsto n_f \mathcal{D}A : x \mapsto n(\downarrow x),$$

for some closure operator n_f on $\mathcal{D}A$. (The material in this paragraph is well known; for example, see Banaschewski and Bruns [2].)

Corollary 30. *Limits in* Z**Frm** *are computed as in* **Set**. *Thus,*

1. *Let A be a Z***Frm***-object. Any intersection of Z***Frm***-subobjects of A is a Z***Frm***-subobject.*
2. *Any Z-frame map $f : A \to B$ has a unique (extremal epi, mono)-factorization $f = i \cdot e$. The extremal epi part e has the feature that $e(A)$ is join-dense in $[fA]$.*



Proof. (1) holds because an intersection of subobjects is a type of limit.

(2) We use several results from [10]. Note that Z**Frm** is wellpowered; Z**Frm** monomorphisms are exactly injections. Thus, any subobject of a Z-frame A may be viewed as a subset the underlying set of A. Also note that Z**Frm** is complete. By [10, Proposition 17.8], the desired factorization is obtained by taking the intersection of the family of all subobjects f factors through. Applying [10, Theorem 34.1], we conclude that this factorization is unique. The image $e(A)$ is join dense in the $[fA]$, because $[fA]$ is the smallest Z-frame containing $e(A)$. □

31. To prove Theorem 32 we need some background from frame theory. A *frame* is a complete lattice in which finite meets distributive over joins. Frame distributivity guarantees that for each a, b, there is a largest element – denoted $a \to b$ such that $a \wedge (a \to b) \leq b$; this implication operation is characterized by

$$c \leq a \to b \iff c \wedge a \leq b.$$

(In fact, a complete lattice with such an implication operation is necessarily a frame.)

Quotients of frames are conveniently described using nuclei; a *nucleus* is a closure operator j which such that $j(a \wedge b) = j(a) \wedge j(b)$. A closure operator j is a nucleus if and only if

$$j(a \to j(b)) = a \to j(b).$$

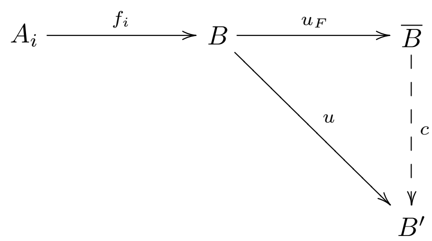
Thus, a subset $S \subseteq A$ of a frame is the image of a nucleus if and only if S is closed under meets and if $s \in S, a \in A$, then $a \to s \in S$.

Nuclei correspond in a bijective order preserving way to frame congruences: if j is a nucleus, then $\{(a, b) : ja = jb\}$ is a frame congruence. If R is a frame congruence, then $j_R(x) = \vee\{x' : (x, x') \in R\}$ is a nucleus. The facts needed are explained in more detail in each of the standard references on frame theory: Johnstone [12] Chapter II, section 2 , the more up to date Johnstone [13] Volume 2, C1.1, Joyal and Tierney [14] Chapter 3, or Vickers [27] Section 6.1.

The proof of the following theorem should come as no surprise. It follows from the discussion in Paragraph 29 and the descriptions of frame congruences given in [14] and [12].

Theorem 32. *Suppose $Z_1 \subseteq Z_2$ are subset selectors. Suppose $F = (f_i : A_i \to B : i \in I)$ is a set of meetsemilattice maps and each A_i is a Z_1-frame. There exists a unique map $u_F : B \to \overline{B}$ satisfying* i(u_F), ii(u_F) *and iii.*

- i(u) codomain(u) *is a Z_2*$\mathbf{Frm}$*-object.*
- ii(u) *$u_F \cdot f_i$ is Z_1-complete.*
- iii *If $u : B \to B'$ is any map satisfying i(u) and ii(u), there is a unique Z_2*$\mathbf{Frm}$*-map $c : \overline{B} \to B'$ such that $c \cdot u_F = u$.*



Proof. Since any map $u : B \to B'$ satisfying i(u) and ii(u) factors through the intersection of all sub-Z_2-frames of B' containing $u(B)$, u_F must have the feature that $u_F(B)$ is join dense in $\overline{B}$, and when proving iii it suffices to consider u where $u(B)$ is join dense in B'. A solution set argument makes the existence of u_F clear, because $2^{|B|}$ is a cardinality bound on B' for maps $f : B \to B'$ where $f(B)$ is join dense in B'. However, a more explicit proof is desirable.

Let us consider the special case of the theorem where Z_2 selects all subsets.

By Paragraphs 31 and 29, u_F may be described as the map

$$u_F : B \to n_F \mathcal{D}B : x \mapsto n_F(\downarrow b),$$

where n_F is the smallest nucleus such that $n_F(\downarrow \vee f_i(S)) = n_F(\downarrow f_i(S))$ for each i and $S \in ZA_i$. We write

$$\mathcal{I}_F = n_F \mathcal{D}B.$$

We describe n_F. Suppose for some i, $J \in \mathcal{I}_F$, $X \in Z_1 A_i$, $b \in B$, and

$$\downarrow b \cap \downarrow f_i(X) \subseteq J.$$

If $n_F \cdot f_i$ preserves Z_1-joins and finite meets, we have

$$b \wedge n_F(\downarrow f_i(X)) = b \wedge n_F(\downarrow f_i(\vee X)).$$

Thus, $\downarrow b \cap \downarrow f_i(\vee X) \subseteq J$.

With the previous paragraph as motivation, define $\mathcal{I}_F$ to be the family of $J \in \mathcal{D}B$ such that for each $b \in B$, $i \in I$, and $X \in ZA_i$

$$\downarrow b \cap \downarrow f_i(X) \subseteq J \implies \downarrow b \cap \downarrow f_i(\vee X) \subseteq J.$$

Evidently, $\mathcal{I}_F$ is closed under intersections, and whenever n is a nucleus such that $n \cdot f_i$ is Z_1complete for each i, $n\mathcal{D}B \subseteq \mathcal{I}_F$. It thus suffices to show that n_F is a nucleus, or equivalently, that if $J \in \mathcal{I}_F$ and $S \in \mathcal{D}B$ then $S \to J \in \mathcal{I}_F$.

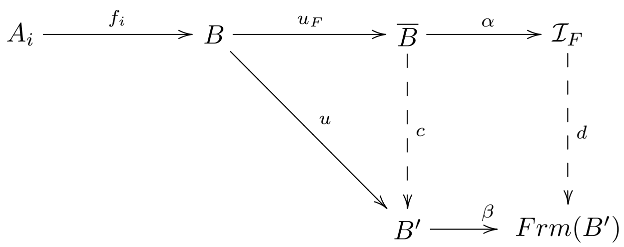
To prove n_F is a nucleus, suppose $J \in \mathcal{I}_F$, $S \in \mathcal{D}B$ and note

$$\begin{aligned} \downarrow b \cap \downarrow f_i(X) \subseteq S \to J \quad &\iff \quad \downarrow b \cap \downarrow f_i(X) \cap S \subseteq J \\ &\iff \quad \{b \wedge f_i(x) \wedge s : x \in X, s \in S\} \subseteq J \\ &\iff \quad \forall s \in S,\, x \in X,\, s \wedge b \wedge f_i(x) \in J \\ &\implies \quad \forall s \in S,\, s \wedge b \wedge (\vee f_i(X)) \in J \\ &\iff \quad b \wedge (\vee f_i(X)) \in S \to J. \end{aligned}$$

This establishes the theorem, in the special case when Z_2 selects all subsets.

Note that if $F = \{\mathrm{id}_{B'}\}$, then $\mathcal{I}_F$ consists of all $J \in \mathcal{D}B'$ such that $S \in Z_1 B' \implies \vee S \in \mathcal{D}B'$. In this case, each $\downarrow b$ for $b \in B'$ is a member of $\mathcal{I}_F$, so u_F is injective. This construction gives the free frame $Frm(B')$ over B'.

Now to prove the general case: we illustrate the maps used with the following diagram.



The map $\alpha \cdot u_F$ is $z : B \to \mathcal{I}_F : x \mapsto n_F(\downarrow x)$; u_F and α are obtained by factoring z through the smallest Z_2-frame containing $z(B)$. If $u : B \to B'$ is any map satisfying i(u) and ii(u), let $\beta : B' \to Frm(B')$ be the map injecting B' in the frame it freely generates (as described in the previous paragraph). By the universal property of z, there is a unique frame map d making $d \cdot z = \beta \cdot u$. Since B' contains $u(B)$, $d(\alpha(\overline{B})) \subseteq B'$. Thus, there is a unique Z_2-frame map c which satisfies the conclusion of the theorem. □

The following was proved in the second to last paragraph of the proof of the preceeding theorem.

Corollary 33. *Suppose $Z_1 \subseteq Z_2$. The forgetful functor $Z_2\mathbf{Frm} \to Z_1\mathbf{Frm}$ has a left adjoint $Z_2Frm_{Z_1}$; the unit of the adjunction is injective.*

Corollary 34. *Z**Frm** is cocomplete.*

Proof. Let $D : I \to Z\mathbf{Frm}$ be any diagram, and $(C, m_i : D(i) \to C)$ its colimit in $\mathcal{M}$. Applying Theorem 32 with $Z = Z_1 = Z_2$ yields $u : C \to \overline{C}$ such that $\overline{C}$ is a Z_1-frame, each $u \cdot m_i$ is Z_1-continuous, and $(\overline{C}, u \cdot m_i)$ has the universal property required. □

A useful consequence of this discussion is that "Z-frame presentations present"; a given set X of generators and E of equations (here an equation should be understood as an element of the $Frm_Z X \times Frm_Z X$ where $Frm_Z X$ is the free-Z-frame over X) there is a Z-frame $Frm_Z(X, E)$ with the expected universal property. To construct $Frm_Z(X, E)$ one assumes without loss of generality that E is a sub-Z-frame of $Frm_Z X \times Frm_Z X$, and computes the coequalizer of the projections $E \to Frm_Z X$.

3. Understanding Subset Selectors

Close relatives of subset selectors have appeared before in frame theory literature (see Dowker and Strauss [9] and Paseka [24]) and there is a large body of literature concerning *subset systems*.

Definition 35. A *subset system* Z is a rule which assigns a family ZA of subsets to each poset A, such that

- for each $x \in A$, $\downarrow x \in ZA$, and
- if $f : A \to B$ is an order preserving map, then $\{f(S) : S \in ZA\} \subseteq ZB$.

Subset systems satisfying (4) of Definition 26 are called *union-complete*. The articles [3] and [22] establish basic properties for subset systems (analogous to our Theorem 32). Venugopalan [26] makes a large list of examples of union complete subset systems.

I wish to emphasize several points regarding subset selectors. First, the rule which selects all subsets of size $< \kappa$ for a regular cardinal κ is one of the "best behaved" subset systems. Imposing too many restrictions on subset systems may reduce the discussion to only these cardinality selectors. These points are elaborated in Subsection 3.1.

Second, there are many examples of subset selectors. The class of subset selectors is closed under arbitrary meets and joins (by Proposition 8). Thus, any class of of subsets of meetsemilattices generates some subset selector. Particular examples of subset selectors are listed in Subsection 3.2

3.1. Subset Selectors and Cardinality. The power set operation $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ can be construed as a covariant functor, which acts on maps by taking $f : A \to B$ to the map

$$\mathcal{P}f : \mathcal{P}A \to \mathcal{P}B : S \mapsto \{f(x) : x \in S\}.$$

There is a clear similarity between the functor $\mathcal{P}$ and the functor $\mathcal{D}$ described above. In fact, $\mathcal{P}$ has a monad structure with

$$\mu^{\mathcal{P}} A(\mathfrak{S}) = \cup\mathfrak{S}$$

and

$$\eta^{\mathcal{P}} A(x) = \{x\}.$$

Thus, the following trivial fact is germane.

Lemma 36. *The only subfunctors of $\mathcal{P}$ are*

- $\mathcal{P}$
- *The functors $[\kappa] \cup 0$ which select all subsets of size less than κ (for any cardinal κ).*
- *The functors $[\kappa]$ which select all non-void subsets of size less than κ.*

$[\kappa]$ is a submonad of $\mathcal{P}$ if and only if κ is regular.

Proof. Let T be a subfunctor of $\mathcal{P}$. Suppose X is a set and a subset $S \in TX$ has cardinality δ. If Y is any nonempty set and $S' \subseteq Y$ has cardinality not exceeding δ, there is a map $X \to Y$ which sends S to S'. So if T selects any set of size δ it selects all sets of size δ.

Let K be the class of all cardinalities k so that $S \in TX \implies |S| < k$. If K is empty, then T selects all subsets of each set. If K is nonempty, it contains a least element κ. By the preceding paragraph, this means that T selects all subsets of size $< \kappa$.

κ is regular if and only if $|\mathfrak{S}| < \kappa$ and $\forall S \in \mathfrak{S}, |S| < \kappa$ implies $| \cup \mathfrak{S}| < \kappa$. □

Let $Z \subseteq \mathcal{D}$ be a subset system. We say Z *preserves surjections* if whenever $f : X \to Y$ is an order preserving surjection

$$ZY = \{\downarrow f(S) : S \in ZX\}.$$

We say a poset $(X, \leq)$ is an *antichain* if $x \leq y \implies x = y$.

Proposition 37. *The only subset systems which preserve surjections are*

- $\mathcal{D}$
- *The functors $[\kappa] \cup 0$ which select all subsets $\downarrow S$ where $|S| < \kappa$.*
- *Then functors $[\kappa]$ which select all nonvoid subsets $\downarrow S$ where $|S| < \kappa$.*

Proof. The proof is analogous to the proof of the preceeding lemma. The only new ingredient is the fact that each poset $(X, \leq)$ is the surjective image of an antichain $(X, =)$. If Z selects some $\downarrow S$ with $|S| = \delta$, then Z selects all subsets of antichains with cardinality δ. Hence Z selects all $\downarrow S$ with $|S| = \delta$. □

Note that union completeness plays a role very similar to regularity of a cardinal; i.e., a subset system is union complete if and only if it is a submonad of $\mathcal{D}$ (construed as a monad over the category of posets).

The property "preservation of surjections" is defined for subset selectors as for subset systems. Recall that the free meetsemilattice FX on a set X consists of all finite formal meets $\sqcap_{i=1}^{n} x_i$ with $x_i \in X$ with the obvious meet operation, i.e., for any two finite sets f_1, f_2, we have $(\sqcap f_1) \sqcap (\sqcap f_2) = \sqcap(f_1 \cup f_2)$. (We denote meets in the free meetsemilattice with $\sqcap$ and in any other meetsemilattice by $\wedge$ for notational clarity in the proof of Proposition 39.) In order to prove Proposition 39 we need the following combinatorial fact:

Theorem 38. [16, Chapter II, Theorem 1.6] *Let κ be any infinite cardinal. Let $\theta > \kappa$ be regular and satisfy $\forall \alpha < \theta$, $\alpha^{<\kappa} < \theta$. Assume $|\mathfrak{A}| \geq \theta$ and $\forall x \in \mathfrak{A}$, $|x| < \kappa$, then there is $\mathfrak{B} \subseteq \mathfrak{A}$ and a fixed set r such that $|\mathfrak{B}| = \theta$ and $\forall x \neq y \in B$, $x \cap y = r$.*

Proposition 39. *Suppose Z is a subset selector which preserves surjections. Then Z is completely determined by which subsets of free meetsemilattices FX it selects. Moreover,*

1. *Each $S \in \mathcal{D}FX$ is $S = \downarrow S_0$ for some antichain S_0.*
2. *If Z selects the empty subset of some meetsemilattice, it selects the empty subset of every meetsemilattice.*
3. *If Z selects any set $\downarrow S$, where S contains at least two incomparable elements a, b, then Z selects every $\downarrow S'$, with S' finite.*
4. *If Z selects any subset $\downarrow S$ with $|S| = \theta$ (uncountable) and S of minimum size with respect to generating $\downarrow S$, i then Z selects all nonempty $\downarrow S'$ where $|S'| \leq \theta$.*
5. *If δ is a singular cardinal, and Z selects each subset $\downarrow S$, where $|S| < \delta$, then Z selects each $\downarrow T$ with $|T| = \delta$.*

Thus, the only possible subset selectors which preserve surjections are $[\kappa]$ and $[\kappa] \cup 0$ (for κ regular) or Z which select only countable subsets.

Proof. The proof of this proposition resembles the proof of Proposition 37; the difference here is that we use free meetsemilattices where we used free posets (i.e., antichains) before.

(1) Let $S \in \mathcal{D}FX$; it suffices to show that $\downarrow S = \downarrow S_{max}$ where S_{max} consists of all maximal elements of S. Let $\sqcap f$ be a finite formal meet in S. Well order the elements of f. By iteratively removing the smallest $x \in f$ such that $\sqcap f \setminus \{x\} \in S$, one obtains a maximal member $\sqcap a$ of S, with $\sqcap a \geq \sqcap f$.

(2) is trivial, because for any meetsemilattices A and B, the constant map $1 : A \to B$ which sends each $a \in A$ to 1_B is a meetsemilattice map, and under this map the image of the empty set is empty.

(3) First note that Z selects all $\downarrow S$ for S finite if and only if $\downarrow \{x, y\} \in ZF\{x, y\}$. For suppose Z selects the set $\downarrow \{x, y\}$ from the meetsemilattice freely generated by $\{x, y\}$. Then Z selects each nonempty downset generated by at most two elements. Now, arguing by induction, suppose Z selects each nonempty downset generated by at most n elements. Thus, $\downarrow \{x_1, \cdots, x_n\} =: U$ and $\downarrow \{x_2, \cdots, x_{n+1}\} =: V$ are in $ZF\{x_1, \cdots x_n, x_{n+1}\}$. Moreover, the set

$$\mathcal{W} := \{S \in ZF\{x_1, \cdots, x_{n+1}\} : S \subseteq U \text{ or } S \subseteq V\}$$

is in $ZZF\{x_1, \cdots, x_{n+1}\}$, because $\mathcal{W}$ is a 2-generated downset. By Definition 26, part 4,

$$\downarrow \{x_1, \cdots, x_{n+1}\} = \cup \mathcal{W} \in ZF\{x_1, \cdots, x_{n+1}\}.$$

By induction, Z selects each nonempty finitely generated downset of each poset.

Now, suppose that $\downarrow S \in ZA$ for some S containing at least two incomparable elements $a \not\leq b, b \not\leq a$. The map

$$g : FA \to A : g(\sqcap f) = \wedge f$$

is a surjective meetsemilattice homomorphism. Since Z preserves surjections, there is a set S' such that $g(S') = S$. S' contains two incomparable finite formal meets $\sqcap f_1, \sqcap f_2$. Thus, there exist $c \in f_1 \setminus f_2$ and $d \in f_2 \setminus f_1$. Define a meetsemilattice map

$$h : FA \to F\{x, y\}$$

by the conditions $h(c) = x, h(d) = y$ and for any $z \in A \setminus \{c, d\}$, $h(z) = x \sqcap y$. (Note that $x \sqcap y$ is the minimum element of $F\{x, y\}$.) Evidently, $\downarrow h(S') = \downarrow \{x, y\}$; thus, $\downarrow \{x, y\} \in ZF\{x, y\}$. This establishes (3).

(4) Z selects each $\downarrow S$ with $|S| = \theta$ if and only if it selects

$$\downarrow \{\sqcap\{\alpha\} : \alpha < \theta\}$$

from $F\theta$. Let $\downarrow S \in ZA$ with $|S| = \theta$, and the feature that there is no T with $|T| < |S|$ and $\downarrow T = \downarrow S$. As above, define

$$g : FA \to A : \sqcap f \mapsto \wedge f.$$

Since Z preserves surjections there is a set $S' \in ZFA$ with cardinality at least θ such that $g(S') = S$. $\downarrow S'$ cannot be generated (as a downset) by a set of cardinality less than θ. Let S'_{max} denote the set of maximal elements of $\downarrow S'$; by (1), $\downarrow S'_{max} = \downarrow S'$, so $|S'_{max}| \geq \theta$. Using Theorem 38, we may select a subset $\{\sqcap f_\alpha : \alpha < \theta\} \subseteq S'_{max}$ with the feature that there is a subset $r \subset A$ such that for any distinct $\alpha, \beta < \theta$, $f_\alpha \cap f_\beta = r$. Recall that the $\sqcap f_\alpha$ are formal meets of finite subsets of A. Assume that for $\alpha \neq \beta$, $f_\alpha \neq f_\beta$. For each $\alpha < \theta$ choose $x_\alpha \in f_\alpha \setminus r$ and define $h : FA \to F\theta$ by $h(\sqcap f_\alpha) = \sqcap\{\alpha\}$ and $h(x) = \sqcap\{0\}$ for $x \in |A| \setminus \{x_\alpha : \alpha < \theta\}$. By construction

$$\downarrow h(S') = \downarrow \{\sqcap\{\alpha\} : \alpha < \theta\} \in ZF\theta.$$

This establishes (4).

The proof of (5) is easy and resembles the first paragraph in the proof of (3), so is left to the reader.

If Z does not select every downset, there is some least upper bound κ on the cardinality $|S|$ of the subsets $\downarrow S$ that Z selects. By (5), κ cannot be a singular cardinal. If $\kappa > \omega_1$ then item (4) from this proposition insures that every subset of cardinality $< \kappa$ is selected. If $\kappa = \omega_0$, then (2) insures that Z selects all finite sets. □

3.2. Examples, Almost Examples and Non Examples. The following subsection will attempt to clarify the concept of subset selector by listing some examples. We use 0 to denote the subset selector defined by

$$0(A) = \{\downarrow x : x \in A\} \cup \{\emptyset\}.$$

We choose the notation 0 because any 0-complete poset has a least element (which we also denote 0) and any 0-complete map preserves least elements. For other subset selectors, we implicitly assume all sets selected are nonempty.

In checking $S, T \in ZA \implies S \cap T \in ZA$ the following identity is useful: for any downward closed sets S and T

$$S \cap T = \{s \wedge t : s \in S,\, t \in T\}.$$

Example 40. We have already seen $[\kappa]$ the subset selector which selects nonempty $\downarrow S$ where $|S| < \kappa$, and κ is a regular cardinal. It is obvious that $[\kappa]$ is a subset selector. Regularity of κ is only used to show that condition (4) of Definition 26 holds.

Example 41. Let α be a regular cardinal. A poset S is α-directed if for each $S_0 \subseteq S$ with $|S_0| < \alpha$, there exists $u \in S$ which is an upper bound for S_0. The rule d_α which selects all $\downarrow S$ for S α-directed is a subset selector. Trivially, $\downarrow x \in d_\alpha A$ for any meetsemilattice A and $x \in A$; if $S, T \in d_\alpha A$ and $S_0 \subseteq S \cap T$ with $|S_0| < \alpha$ there are upper bounds $t \in T$ and $s \in S$ for S_0. Then $t \wedge s$ is an upper bound for S_0 in $S \cap T$. Any image of an α-directed set is α-directed. If $\mathfrak{S}$ is an α-directed family of α-directed sets, and $S_0 \subseteq \cup\mathfrak{S}$, then there is $S \in \mathfrak{S}$ with $S_0 \subseteq S$. Thus, $\cup\mathfrak{S}$ is α-directed.

The class of subset selectors forms a complete lattice. Thus, $[\kappa] \cap d_\alpha$, $[\kappa] \vee d_\alpha$ (with $\kappa < \alpha$), and $([\kappa] \vee d_\alpha) \cap [\lambda]$ (with $\kappa < \alpha < \lambda$) are subset selectors. With some difficulty, one may verify that $[\kappa] \vee d_\alpha$ consists of all $\downarrow S$ where S has the feature that each subset with cardinality less than α is bounded by a subset of S with cardinality less than κ. (Proof of this fact is omitted because the space required to give a detailed argument is greater than the importance of the result.) Hence, the combinations of $[\kappa]$, $[\lambda]$ and d_α listed above are distinct subset selectors.

At present, the author is not aware of any other subset selectors on $\mathcal{M}$. However, the main results of this paper on subset selectors (Lemma 22, Paragraph 23, Proposition 27, Theorem 32 and it corollaries) hold with obvious and slight changes in their proofs if instead of subset selectors on $\mathcal{M}$ we consider subset selectors on

- $\mathcal{M}'$ – the category of meetsemilattices without top element. The objects are posets in which each pair of elements has a meet and the maps preserve binary meets.
- $\mathcal{M}_0$ – the category of meetsemilattices with 0. Objects are meetsemilattices A with a distinguished constant 0, satisfying $a \wedge 0 = 0$ for all $a \in A$. Maps preserve finite meets (in particular 1) and 0.
- The categories of κ-complete meetsemilattices, with or without 0 or 1, and maps which preserve κ-meets and distinguished elements. (Note: the "explicit" proof of Theorem 32 given in this article does not work. However, a solution set argument as given in [3] still establishes the proposition.)

42. A similar theory of arbitrary submonads of completely distributive complete lattice monads might also be developed – using two subset systems (one indicating when joins exist and another when meets exist) but there are some technical problems to be overcome. The author's thesis [28] aimed to understand these submonads. One needs to use monad distributive laws. Also [28] was unable to characterize F-algebras for F an arbitrary submonad of the completely distributive lattice monad.

We now enumerate some almost examples of subset selectors, i.e., subset selectors over $\mathcal{M}'$ and $\mathcal{M}_0$.

Example 43. The rule for which A consists of all downward closed subsets $S \subseteq A$ which have an upper bound in A is a subset selector on $\mathcal{M}'$.

Example 44. Say that x, y are disjoint if $x \wedge y = 0$. Let D be the rule (considered over $\mathcal{M}_0$) which selects pairwise disjoint subsets. D does not satisfy condition (4) of Definition 26, but it has a monadic closure. It is possible for a meetsemilattice to have joins of all pairwise disjoint sets without it being κ-complete or α-directed complete, e.g., consider a chain which is not κ-complete or α-directed complete.

Example 45. Let us say x and y are a-disjoint if $x \wedge y \leq a$. The rule J for which JA consists of all $\downarrow S$ such that there is an $a \in A$ with each distinct pair in S a-disjoint is a rule satisfying (1), (2) and (3) of Definition 26. Therefore, there is a monadic closure $\overline{J}$ of J. Considered over $\mathcal{M}'$, J-completeness is different from κ-completeness and α-directed completeness, because some well-ordered sets (κ^+ and α^+ considered as initial ordinals respectively) are J-complete without being α-directed complete or κ-complete.

It may be instructive to give some examples of rules which do not satisfy (3) of Definition 26: the rule which selects all subsets of size larger than κ, the rule which selects all antichains, the rule which selects all members of any class which is not closed under surjections.

References

[1] Banaschewski B., On pushing out frames,*Comment. Math. Univ. Carolin.* **31** (1990), 13-21.

[2] Banaschewski B., Bruns B., Categorical characterization of the MacNeille Completion, *Archiv d. Math.*, **18** (1967), 369-377.

[3] Banaschewski B., Nelson E., Completions of partially ordered sets, *Siam J. Comput.* **11** no. 3 (1982), 521-528.

[4] Banaschewski B., Another look at the localic Tychonoff Theorem, *Comment. Math. Univ. Carolinae* **29** (1988) 647-656.

[5] Barr M., Coequalizers and free triples, Math. Z.,**116** (1970) 307-322.

[6] Barr M., Wells C.,*Toposes, triples, and theories*, Springer-Verlag, Berlin-New York (1985).

[7] Beck J., Distributive laws, in: Seminar on Triples and Categorical Homology Theory, *Lecture Notes in Mathematics* **80**, Springer-Verlag, Berlin-New York (1969) 119-140.

[8] Borceux F., *Handbook of categorical algebra*, (3 volumes), Cambridge University Press, Cambridge (1994).

[9] Dowker C. H., Strauss D., Sums in the category of frames, Houtson J. Math., 3 (1973) 17-32.

[10] Herrlich H., Strecker G., *Category theory*, Heldermann-Verlag, Berlin (1979).

[11] Johnstone P. , Vickers S., Preframe presentations present, Category theory (Como, 1990), 193-212, *Lecture Notes in Math.*, 1488, Springer, Berlin (1991).

[12] Johnstone P., *Stone Spaces*, Cambridge studies in advanced mathematics, Cambridge (1982).

[13] Johnstone P., *Sketches of an Elephant: A Topos Theory Compendium* (2 volumes), Clarenden Press, Oxford (2002).

[14] Joyal A., Tierney M., *An extension of the Galois theory of Grothendieck.*, Mem. Amer. Math. Soc., no. 23 (1980).

[15] Kůrkov-Pohlov V., Koubek V., When a generalized algebraic category is monadic, Comment. Math. Univ. Carolinae **15** (1974), 577–587.

[16] Kunen K., *Set Theory: an introduction to independence proofs*, Elsevier Studies in Logic and The Foundations of Mathematics, 7th impression (1999).
[17] Linton F. E. J., Coequalizers in categories of algebras, in: Seminar on triples and categorical homology theory, *Lecture Notes in Mathematics* **80**, Springer-Verlag, Berlin-New York (1969) 75-90.
[18] MacLane S., Categories for the working Mathematician, *Graduate Texts in Mathematics* **5**, Springer-Verlag, Berlin-New York (1971).
[19] Madden J. J., κ-frames, *Journal of Pure and Applied Algebra*, **70** (1991) 107-127.
[20] Madden J. J., Molitor A., Epimorphisms of frames, *J. Pure Appl. Algebra* **70** (1991), 129-132.
[21] Manes E. G., *Algebraic Theories*, *Graduate Texts in Mathematics* **26**, Springer-Verlag, Berlin-New York (1976).
[22] Meseguer J., Order completion monads, *Algebra Universalis*, **16** (1983) 63-82.
[23] Marmolejo F., Rosebrugh R. D., Wood R. J., A basic distributive law. *J. Pure Appl. Algebra* **168** (2002), 209-226.
[24] Paseka J., Covers in generalized frames, in: *General Algebra and Ordered Sets (Horni Lipova 1994)*, 84-99, Palacky Univ. Olomouc, Olomouc.
[25] Thatcher J. W., Wright J. B. & Wagner E. G., A uniform approach to inductive posets and inductive closure, *Theoretical Computer Science*, **7** (1978) 57-77.
[26] Venugopalan P., *Union complete subset systems*, Houston J. Math., **14** No. 4 (1988) 583-600.
[27] Vickers S., *Topology Via Logic*, Cambridge Tracts in Theoretical Computer Science, Cambridge U. Press (1989).
[28] Zenk E. R., Subset systems and generalized distributive lattices, Doctoral Dissertation, University of Florida, (2004).

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