Math 274 Identities (1.2)

The combinatorial argument for proving identities usually involves showing both sides of the identities count the same set. The difficulty is devising an appropriate set.

In this lecture, we introduce some more models to prove the identities involving $\binom{n}{k}$.

1. Lattice paths and Pascal's Triangles

A lattice point is a vector with integer coordinates. A lattice step changes one coordinate by one. A lattice path moves from one lattice point to another in which each step increases one coordinate.

There is a bijection between planar lattice paths and binary lists.

(1) Prove the following identities:

(i) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

(ii) $n\binom{n}{k} = n\binom{n-1}{k-1}$, and more general $\binom{k}{l}\binom{n}{k} = \binom{n}{l}\binom{n-l}{k-l}$.

(iii) $\sum_{k=0}^{n}\binom{k}{r} = \binom{n+1}{r+1}$, and more general, $\sum_{k=-m}^{n}\binom{m+k}{r}\binom{n-k}{s} = \binom{m+n+1}{r+s+1}$.

(iv) $\sum_k \binom{n}{k}^2 = \binom{2n}{n}$, and more general, $\sum_k \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$.

(v) $\sum_k r^k\binom{n}{k} = (r+1)^n$.

Remark: this identity can be used to compute $\sum_{i=1}^{n} i^k$.

2. Delannoy Numbers

(2) **Delannoy number** $d_{m,n}$ is the number of paths from $(0,0)$ to (m,n) such that each move is by one of $\{(1,0),(0,1),(1,1)\}$. The number of the form $d_{n,n}$ are the **Central Delannoy numbers**.

$$d_{m,n} = \sum_k \binom{m}{k}\binom{n+k}{m}.$$

Proof: Suppose that there are k horizontal moves, then there are $m-k$ diagonal moves. Thus there are $m+n-(m-k) = n+k$ moves, since horizontal and vertical moves add one to the coordinate sum and diagonal moves add 2. Then we place the moves increase the horizontal coordinate and place the horizontal moves.

(3) The **Hamming ball** of radius m around the origin is the s set of all integer n-tuples such that the absolute values of the coordinates sum to at most m. Prove that the size of this set is exactly $\sum_k \binom{n}{k}\binom{m}{k}2^k$.

Proof: Consider the non-zero coordinates.

(4) Prove that

$$\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{n}{k}\binom{m}{k} 2^k.$$

First Proof: (bijection) Let A be the set of Delannoy paths to (m, n), and let B be the Hamming ball of radius m in Z_n. Construct a bijection $f : A \to B$ in the following way:

Consider a Delannoy path P first. For a sub-path from $(*, i-1)$ to $(*, i)$, let $|b_i|$ be the number of horizontal or diagonal steps. The sign of b_i is positive if the last step is vertical, negative if it is diagonal. Let $f(P) = b = (b_1, b_2, \ldots, b_n)$.

From an n-tuple with $\sum_{i=1}^{n} |b_i| \leq m$, we can reverse the previous process.

Second Proof: Let M and N be sets with sizes m and n. let S be the family of ordered pairs (X, Y) such that X is a subset of M and Y is an m-subset of $N \cup X$. Both sides count S.

LHS: let $k = |X|$.

RHS: let $k = |Y \cap N|$. Count Y first, then form X from Y.