LECTURE 3: EIGENVECTORS AND EIGENVALUES I

1. *Introductory matters and administrative announcements*

1.1. *Reading*: Shankar, pp. 29-43. Homework *for Friday*, *Sept 3rd*: Exercises 1.6.4, 1.7.1, 1.8.2, 1.8.3. *Homework for Monday, Sept 6th*: Exercises 1.8.10, 1.8.11, 1.9.1, 1.9.2.

1.2. *Colloquia*: VINSE Colloquium, Jun Kono (Rice), "Controlling the Metallicity of Carbon Nanotubes," 5326 Stevenson, Wednesday 4 p.m. Physics Colloquium Thursday 4 p.m., 4327 Stevenson, P. G. Hansen (Michigan State University), "Correlated Wave Functions." Better refreshments are this afternoon.

1.3. *Objectives for today*: We have now become acquainted with the various properties of vectors, inner products and linear operators. Today the focus is on three topics:

1.3.1. Unitary and Hermitian operators, analogous to complex and real numbers, as in the following: $UU^\dagger = I \Leftrightarrow uu^* = 1,\ H = H^\dagger \Leftrightarrow h = h^*$

1.3.2. The general statement of the eigenvalue problem.

1.3.3. Application to eigenvalues of Hermitian operators, and diagonalization.

2. *Reprise*: Examples of linear operators drawn from optics, operating on vectors [E_x, E_y] representing the electric field of a wave propagating in the z-direction.

2.1. The linear polarizer with transmission angle ϑ with respect to the x-axis.

$$R\left(\vartheta\hat{k}\right) = \begin{bmatrix} cos^2\vartheta & cos\vartheta \cdot sin\vartheta \\ cos\vartheta \cdot sin\vartheta & cos^2\vartheta \end{bmatrix}$$

2.2. The phase shifter (typical of anisotropic materials with two indices of refraction)

$$R = \begin{bmatrix} e^{i\varepsilon_x} & 0 \\ 0 & e^{i\varepsilon_y} \end{bmatrix}$$

2.3. The polarization rotator (derivation assigned for homework today)

$$R\left(\beta\hat{k}\right) = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

3. *The eigenvalue problem I*. Stated simply, the *eigenkets* of the linear operator Ω are those kets that satisfy $\Omega|V\rangle = \omega|V\rangle$, where ω is a complex scalar.

3.1. In order for this equation to be true, we must have $det\left[\tilde{\Omega} - \omega\tilde{I}\right] = 0$.

3.2. Now we develop the eigenvalue equation in a specific basis so that we can use the matrix representation and properties of the operator Ω and the identity operator I:

$$\langle i|\left(\tilde{\Omega} - \omega\tilde{I}\right)|V\rangle = \langle i|\left(\tilde{\Omega}\tilde{I} - \omega\tilde{I}\right)|V\rangle = \langle i|\left(\tilde{\Omega}\sum_j |j\rangle\langle j| - \omega\sum_j |j\rangle\langle j|\right)|V\rangle = \sum_j \left(\Omega_{ij} - \omega\delta_{ij}\right)v_j = 0$$

Carrying out the indicated matrix multiplication, we find that the eigenvalue condition on the determinant produces a *characteristic equation* with n roots, not necessarily real and not necessarily distinct

$$\sum_{m=0}^{n} c_m \omega^m \equiv P^n(\omega) = 0$$

3.3. *Example*: To find the eigenvectors and eigenvalues of the rotation matrix $R(\pi\mathbf{i}/2)$, we exhibit the matrix in an appropriate (Cartesian) basis, find the eigenvalues and then substitute into the eigenvalue equation to find assigned the basis kets:

$$\tilde{R}|1\rangle = |1\rangle,\ \tilde{R}|2\rangle = |3\rangle,\ \tilde{R}|3\rangle = -|2\rangle \Rightarrow \tilde{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

3.4. Then we use the definition to find the *characteristic equation* that leads us to the set of linear equations that defines the eigenvalues:

$$\det\begin{bmatrix} 1-\omega & 0 & 0 \\ 0 & -\omega & -1 \\ 0 & 1 & -\omega \end{bmatrix} = (1-\omega)\left(\omega^2+1\right) = 0 \Rightarrow \omega = 1,\ \pm i$$

The eigenvectors are found by substituting particular eigenvalues into this equation:

$$\begin{bmatrix} 1-1 & 0 & 0 \\ 0 & 0-1 & -1 \\ 0 & 1 & 0-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_2 - x_3 \\ x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow |\omega_1\rangle = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$$

The bottom two equations imply that the only non-zero component can be x_1. So choose the normalized eigenvector [1,0,0]. Likewise for the other two eigenvalues.

4. Now we have several important theorems regarding Hermitian and unitary operators, which play critical roles as representations of physical observables. The theorems are

4.1. The eigenvalues of a Hermitian operator are real.

$$\langle V|\tilde{H}|V\rangle = \langle V|h|V\rangle = h\langle V|V\rangle = \langle V|\tilde{H}^{\dagger}|V\rangle = \langle V|h^*|V\rangle = h^*\langle V|V\rangle \Rightarrow h = h^*$$

4.2. For every Hermitian operator Ω, there exists at least one basis composed of its orthogonal and normalized eigenvectors. In this basis, the operator Ω is diagonal, and the eigenvalues are the diagonal entries. Proof: read carefully pp. 36-37!

4.3. The eigenvalues of unitary operators are complex numbers with unit modulus, and the eigenvectors are mutually orthogonal (assuming no degeneracy).

$$\langle j|\tilde{U}^{\dagger}\tilde{U}|i\rangle = \langle j|u_j^* u_i|i\rangle = u_j^* u_i\langle j|i\rangle = |u_i|^2,\ i = j \ \ or \ \ 0,\ i \neq j$$

5. Taking all these properties together, we can show that **If $\tilde{\Omega}$ is Hermitian, there exists a matrix $\tilde{U}$** (built from eigenvectors of W) **such that $\tilde{U}^{\dagger}\tilde{\Omega}\tilde{U}$ is diagonal.**