Math 274 Homework Two–solution

(1) *Prove bijectively that the Catalan number C_n counts the arrangements of $2n$ people in 2 rows of n such that heights increases in each row and column.*

we build a bijection between the arrangements and ballot paths. Suppose that people are labeled with $u_1, u_2, \ldots, u_n$ and $l_1, l_2, \ldots, l_n$ such that for $1 \leq i < j \leq n$, $u_i < u_j$, $l_i < l_j$, and $u_i < l_i$. For the k-th step, it is horizontal if $l_k \leq u_{k+1}$, vertical otherwise. I omit the argument to show this is a bijection.

(2) *A fair coin is flipped $2n$ times, keeping a running total of heads and tails. Compute the probability that the lead changes, given that they are tied at the end.*

The total number of plays is $\binom{2n}{n}$, and among them, C_n of them head always leads, and C_n of them tail always leads. So the probability that the lead changes is $\frac{\binom{2n}{n}-2C_n}{\binom{2n}{n}} = \frac{n-1}{n+1}$.

(3) *Prove the identity below by counting in two ways.*

$$\sum_k \frac{1}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k} = \binom{2n+1}{n}$$

Let's count the lattice paths from $(0,0)$ to $(n+1,n)$. This may divided into cases depending on the first time the path touch the line $y = x$. When a path first touches the $y = x$ at $(k+1,k+1)$, the first step is horizontal and the last step is vertical. Thus it is the number of ballot paths from $(1,0)$ to $(k+1,k)$, and we have C_k of which. There are $\binom{2n-2k}{n-k}$ of paths from $(k+1,k)$ to $(n+1,n)$.

(4) *Place $2n$ points on a circle. Prove bijectively that the number of ways to pair the points by drawing noncrossing chords equals the number of ballot lists of length $2n$. (Page 58, 1.3.19)*

Label the points as $1, 2, \cdots, 2n$ as their order appearing in the circle. For a pairing with no crossing chords, we can form a ballot $2n$-list $a_1 a_2 \cdots a_{2n}$: $a_k = 1$ if k is unpaired from $1, 2, \ldots, k-1$, 0 otherwise. Note that the head of a chord always appears before its tail, this is indeed a ballot list. Conversely we can form a pairing with noncrossing chords from a ballot $2n$-path.

(5) *Obtain a recurrence for the number of pairings of $2n$ people.*

Let p_n be the number of pairings of $2n$ people. Then for the first person, there are $2n-1$ ways to pair it up, and there are p_{n-1} ways to pair up the remaining people. So $p_n = (2n-1)p_{n-1}$.

(6) *Prove bijectively that*

$$\hat{F}_{n+m} = \hat{F}_n \hat{F}_m + \hat{F}_{n-1} \hat{F}_{m-1}.$$

Conclude that for each positive integer k, $\hat{F}_{n-1}$ divides $\hat{F}_{kn-1}$.

$\hat{F}_{n+m}$ counts ways to tile $1 \times (m+n)$ board with 1×1 and 1×2 blocks. Depending on whether the first $1 \times m$ board forms a tiling, we have two cases: $\hat{F}_n \hat{F}_m$ ways to make it a tiling, and $\hat{F}_{n-1} \hat{F}_{m-1}$ ways to fail since the first $1 \times (m-1)$ and the last $1 \times (n-1)$ form tiles.