

STRONG SOLIDITY OF GROUP FACTORS FROM LATTICES IN $\mathrm{SO}(n, 1)$ AND $\mathrm{SU}(n, 1)$

THOMAS SINCLAIR

ABSTRACT. We show that the group factors $L\Gamma$, where Γ is an ICC lattice in either $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$, $n \geq 2$, are strongly solid in the sense of Ozawa and Popa [13]. This strengthens a result of Ozawa and Popa [14] showing that these factors do not have Cartan subalgebras.

INTRODUCTION

In their breakthrough paper [13], Ozawa and Popa brought new techniques to bear on the study of free group factors which allowed them to show that these factors possess a powerful structural property, what they called “strong solidity.”

Definition 0.1 (Ozawa–Popa [13]). A II_1 factor M is *strongly solid* if for any diffuse amenable subalgebra $P \subset M$ we have that $\mathcal{N}_M(P)''$ is amenable.

As usual, $\mathcal{N}_M(P) = \{u \in \mathcal{U}(M) : uPu^* = P\}$ denotes the *normalizer* of P in M . It can be seen that every nonamenable II_1 subfactor of a strongly solid II_1 factor is non-Gamma, prime and has no Cartan subalgebras. Thus, Ozawa and Popa’s result broadened and offered a unified approach to the two main results on the structure of free group factors hitherto known: Voiculescu’s [29] pioneering result, which showed that the free group factors $L\mathbb{F}_n$, $2 \leq n \leq \infty$, have no Cartan subalgebras, and Ozawa’s [12] seminal work on “solid” von Neumann algebras, which showed that every nonamenable II_1 subfactor of a free group factor is non-Gamma and prime. Moreover, they exhibited the first, and so far only, examples of II_1 factors with a unique Cartan up to unitary conjugacy; namely, the group-measure space constructions of free ergodic profinite actions of groups with property $(\mathrm{HH})^+$ [14]; e.g., nonamenable free groups. This improved on the ground-breaking work of Popa [20], which gave examples of II_1 factors with a unique “HT-Cartan” subalgebra up to unitary equivalence; e.g., $L(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}))$.

By incorporating ideas and techniques of Peterson [17], Ozawa and Popa [14] were later able to extend the class of strongly solid factors to, in particular, all group factors of ICC lattices in $\mathrm{PSL}(2, \mathbb{R})$ or $\mathrm{PSL}(2, \mathbb{C})$. Other examples of strongly solid factors were subsequently constructed by Houdayer [9] and by Houdayer and Shlyakhtenko [10].

By a *lattice* we mean a discrete subgroup $\Gamma < G$ of some Lie group with finitely many connected components such that G/Γ admits a regular Borel probability measure invariant

Date: December 9, 2010.

2010 Mathematics Subject Classification. 46L10.

Key words and phrases. strong solidity, lattices.

under left translation by G . The main goal of this paper will be to demonstrate the following result:

Theorem 0.2. *If Γ is an ICC lattice in $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$, then $L\Gamma$ is strongly solid.*

These factors are already known by the work of Ozawa and Popa [14] to have no Cartan subalgebras. Since $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$ are simple Lie groups with finite center, Borel’s density theorem via Theorem 6.5 in [5] shows that every $\gamma \in \Gamma$ which is not in the center of G has infinite Γ -conjugacy class, so examples of ICC lattices abound. In the $\mathrm{SO}(n, 1)$ case, the restriction of the lattice subgroup Γ to the connected component of the identity $\mathrm{SO}(n, 1)_0$ is always ICC, $\mathrm{SO}(n, 1)_0$ having trivial center, and all results in our paper will hold for these groups as well. In particular, we have that $\mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{SO}(2, 1)_0 \cong \mathrm{SU}(1, 1)$ and $\mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{SO}(3, 1)_0$, so Theorem 0.2 recovers the main result in Ozawa–Popa [14]. Finally, notice that if G is a Lie group with finite center and finitely many connected components which is locally isomorphic to $\mathrm{SO}(n, 1)$, then it is a finite-to-one covering of—hence, a finite extension of— $\mathrm{SO}(n, 1)_0$. Cohomological induction, combined with the techniques below, will then be sufficient to show that the group von Neumann algebra of any ICC lattice in such a Lie group is also strongly solid.

The proof follows the same strategy as Ozawa and Popa’s in [13, 14]. Though, instead of working with closable derivations, we use a natural one-parameter family of deformations first constructed by Parthasarathy and Schmidt [15]. The derivations Ozawa and Popa consider appear as the infinitesimal generators of these deformations (so, the approaches are largely equivalent), but by using the deformations we avoid some of the technical issues which arise when working with derivations.

The main difficulty in obtaining Theorem 0.2 for lattice factors in $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(m, 1)$ when $n \geq 4$ or $m \geq 2$ is that the bimodules which admit good deformations/derivations are themselves too weak to allow one to deduce the amenability of the normalizer algebra e.g., strong solidity. However, sufficiently large tensor powers of these bimodules can be used to deduce strong solidity. Unfortunately, derivation techniques perturb the original bimodules slightly, and the behavior of tensor powers of the perturbed bimodules becomes unclear. To circumvent this problem, we first notice that Ozawa and Popa’s techniques actually allow one to deduce a kind of relative amenability of the normalizer subalgebra with respect to the bimodule, given in terms of an “invariant mean”. We then use a result of Sauvageot [25] to obtain from the invariant mean an almost invariant sequence of vectors in the bimodule. Since the property of having an almost invariant sequence of vectors is stable under taking tensor powers, we are able to transfer relative amenability to a large tensor power of the bimodule in order to deduce amenability of the normalizer algebra.

We remark that as a corollary to the techniques used in the proof of Theorem 0.2, we are able to strengthen a result of Houdayer [9] on free product group factors admitting no Cartan subalgebras.

Theorem 0.3. *Let Γ be a nonamenable, countable, discrete group which has the complete metric approximation property (Definition 1.6). If $\Gamma \cong \Gamma_1 * \Gamma_2$ decomposes as a non-trivial free product, then $L\Gamma$ has no Cartan subalgebras. Moreover, if $N \subset L\Gamma$ is a nonamenable*

subfactor which has a Cartan subalgebra, then there exists projections p_1, p_2 in the center of $N' \cap L\Gamma$ such that $p_1 + p_2 = 1$ and unitaries $u_1, u_2 \in \mathcal{U}(M)$ such that $u_i N p_i u_i^* \subset L\Gamma_i \subset L\Gamma$, $i \in \{1, 2\}$.

1. PRELIMINARIES

We collect in this section the necessary definitions, concepts and results needed for the proofs of Theorems 0.2 and 0.3.

1.1. Representations, Correspondences and Weak Containment. Let Γ be a countable discrete group and π, ρ be unitary representations of Γ into separable Hilbert spaces \mathcal{H}_π and \mathcal{H}_ρ , respectively.

Definition 1.1. We say that ρ is *weakly contained* in π if for any $\varepsilon > 0$, $\xi \in \mathcal{H}_\rho$ and any finite subset $F \subset \Gamma$, there exist vectors $\xi'_1, \dots, \xi'_n \in \mathcal{H}_\pi$ such that $|\langle \rho(\gamma)\xi, \xi \rangle - \sum_{i=1}^n \langle \pi(\gamma)\xi'_i, \xi'_i \rangle| < \varepsilon$ for all $\gamma \in F$.

A representation π is said to be *tempered* if it is weakly contained in the left*-regular representation, and *strongly ℓ^p* [27] if for any $\varepsilon > 0$, there exists a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that for all $\xi, \eta \in \mathcal{H}_0$ the matrix coefficient $\langle \pi(\gamma)\xi, \eta \rangle$ belongs to $\ell^{p+\varepsilon}(\Gamma)$. By a theorem of Cowling, Haagerup and Howe [6], a representation which is strongly ℓ^2 is tempered. As was pointed out in [27], applying standard Hölder estimates to the matrix coefficients, we obtain that if π is strongly ℓ^p for some $p \geq 2$, then for all $n > p/2$, $\pi^{\otimes n}$ is strongly ℓ^2 , hence tempered.

In the theory of von Neumann algebras, correspondences (also called Hilbert bimodules) play an analogous role to unitary representations in the theory of countable discrete groups. For von Neumann algebras N and M , recall that an N - M correspondence is a *-representation π of the algebraic tensor $N \odot M^o$ into the bounded operators on a Hilbert space \mathcal{H} which is normal when restricted to both N and M^o . We will denote the restrictions of π to N and M^o by π_N and π_{M^o} , respectively. When the N - M correspondence π is implicit for the Hilbert space \mathcal{H} , we will use the notation $x\xi y$ to denote $\pi(x \otimes y^o)\xi$, for $x \in N$, $y \in M$ and $\xi \in \mathcal{H}$.

Definition 1.2. Let $\pi : N \odot M^o \rightarrow \mathbb{B}(\mathcal{H}_\pi)$, $\rho : N \odot M^o \rightarrow \mathbb{B}(\mathcal{H}_\rho)$ be correspondences. We say that ρ is *weakly contained* in π if for any $\varepsilon > 0$, $\xi \in \mathcal{H}_\rho$ and any finite subsets $F_1 \subset N$, $F_2 \subset M$, there exist vectors $\xi'_1, \dots, \xi'_n \in \mathcal{H}_\pi$ such that $|\langle x\xi y, \xi \rangle - \sum_{i=1}^n \langle x\xi'_i y, \xi'_i \rangle| < \varepsilon$ for all $x \in F_1$, $y \in F_2$.

There is a well-known functor from the category whose objects are (separable) unitary representations of Γ and morphisms weak containment to the one of $L\Gamma$ - $L\Gamma$ correspondences and weak containment, cf. [22], which translates the representation theory of Γ into the theory of $L\Gamma$ - $L\Gamma$ correspondences. The construction is as follows. Given $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$ a unitary representation, let \mathcal{H}^π be the Hilbert space $\mathcal{H}_\pi \otimes \ell^2\Gamma$. Then, the maps $u_\gamma(\xi \otimes \eta) = \pi(\gamma)\xi \otimes u_\gamma\eta$, $(\xi \otimes \eta)u_\gamma = \xi \otimes (\eta u_\gamma)$ extend to commuting normal representations of $L\Gamma$ and $(L\Gamma)^o$ on \mathcal{H}^π : the former by Fell's absorption principle, the latter trivially. This

functor is well-behaved with respect to tensor products; i.e., $\mathcal{H}^{\pi \otimes \rho} \cong \mathcal{H}^\pi \otimes_{L\Gamma} \mathcal{H}^\rho$ as $L\Gamma$ - $L\Gamma$ correspondences for any unitary Γ -representations π and ρ . We refer the reader to [1, 19] for the theory of tensor products of correspondences and the basic theory of correspondences in general.

For a II_1 factor M there are two canonical correspondences: the *trivial correspondence*, $L^2(M)$ with M acting by left and right multiplication, and the *coarse correspondence*, $L^2(M) \otimes L^2(\bar{M})$ with M acting by left multiplication of the left copy of $L^2(M)$ and right multiplication on the right copy. When $M = L\Gamma$ for some countable discrete group, the trivial and coarse correspondences are the correspondences induced respectively by the trivial and left regular representations of Γ .

1.2. Cocycles and the Gaussian Construction. In this section, \mathcal{H} will denote a *real* Hilbert space which we will fix along with a orthogonal representation $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H})$ of some countable discrete group Γ .

Definition 1.3. A cocycle is a map $b : \Gamma \rightarrow \mathcal{H}$ satisfying the cocycle relation

$$b(\gamma\gamma') = b(\gamma) + \pi(\gamma)b(\gamma'), \text{ for all } \gamma, \gamma' \in \Gamma.$$

Given π and \mathcal{H} , there is a canonical standard probability space (X, μ) and a canonical measure-preserving action $\Gamma \curvearrowright^\sigma (X, \mu)$ such that there is a Hilbert space embedding of \mathcal{H} into $L^2_{\mathbb{R}}(X, \mu)$ intertwining π and the natural representation induced on $L^2_{\mathbb{R}}(X, \mu)$ by σ . This is known as the *Gaussian construction*, cf. [18] or [26]. It is well-known that the natural Γ representation σ_0 on $L^2_0(X, \mu) = L^2(X, \mu) \ominus \mathbb{C}1_X$ inherits all “stable” properties from π , cf. [18]. In particular, $\sigma_0^{\otimes n}$ is tempered if and only if $\pi^{\otimes n}$ is tempered for any $n \geq 1$.

It was discovered by Parthasarathy and Schmidt [15] that cocycles also fit well into the framework of the Gaussian construction, inducing one-parameter families of deformations (i.e., cocycles) of the action σ . To be precise:

Theorem 1.4 (Parthasarathy–Schmidt [15]). *Let $b : \Gamma \rightarrow \mathcal{H}$ be a cocycle, then there exists a one-parameter family $\omega_t : \Gamma \rightarrow \mathcal{U}(L^\infty(X, \mu))$, $t \in \mathbb{R}$ such that:*

$$(1.1) \quad \omega_t(\gamma\gamma') = \omega_t(\gamma)\sigma_\gamma(\omega_t(\gamma')), \text{ for all } \gamma, \gamma' \in \Gamma; \text{ and}$$

$$(1.2) \quad \int \omega_t(\gamma) d\mu = \exp(-(t\|b(\gamma)\|)^2), \text{ for all } \gamma \in \Gamma.$$

1.3. Weak Compactness and the CMAP.

Definition 1.5 (Ozawa–Popa [13]). Let (P, τ) be a finite von Neumann algebra equipped with a trace τ , and $G \curvearrowright^\sigma P$ be an action of a group G on P by τ -preserving $*$ -automorphisms. We say that the action σ is *weakly compact* if there exists a net of unit vectors $(\eta_k) \in L^2(P \otimes \bar{P}, \tau \otimes \bar{\tau})_+$ such that:

- (1) $\|\eta_k - (v \otimes \bar{v})\eta_k\| \rightarrow 0$, for all $v \in \mathcal{U}(P)$;
- (2) $\|\eta_k - (\sigma_g \otimes \bar{\sigma}_g)\eta_k\| \rightarrow 0$, for all $g \in G$; and
- (3) $\langle (x \otimes 1)\eta_k, \eta_k \rangle = \tau(x) = \langle (1 \otimes \bar{x})\eta_k, \eta_k \rangle$, for all $x \in P$.

Definition 1.6. A II_1 factor M is said to have the *complete metric approximation property* (CMAP) if there exists a net (φ_i) of finite-rank, normal, completely bounded maps $\varphi_i : M \rightarrow M$ such that $\limsup \|\varphi_i\|_{\text{cb}} \leq 1$ and such that $\|\varphi_i(x) - x\|_2 \rightarrow 0$, for all $x \in M$.

If Γ is an ICC countable discrete group, then $L\Gamma$ has the CMAP if and only if the Cowling-Haagerup constant of Γ , $\Lambda_{\text{cb}}(\Gamma)$, equals 1, and if Γ is a lattice in G , then $\Lambda_{\text{cb}}(\Gamma) = \Lambda_{\text{cb}}(G)$, cf. §12.3 of [3] and [8].

Theorem 1.7 (Ozawa–Popa, Theorem 3.5 of [13]). *Let M be a II_1 factor which has the CMAP. Then for any diffuse amenable $*$ -subalgebra $A \subset M$, $\mathcal{N}_M(A)$ acts weakly compactly on A by conjugation.*

2. AMENABLE CORRESPONDENCES

Definition 2.1 (Anantharaman-Delaroche [1]). An N - M correspondence \mathcal{H} is called (*left*) *amenable* if $\mathcal{H} \otimes_M \bar{\mathcal{H}}$ weakly contains the trivial N - N correspondence.

The concept of amenability for correspondences is the von Neumann algebraic analog of the concept of amenability of a unitary representation of a locally compact group due to Bekka [2]. As was observed by Bekka, amenability of the representation π is equivalent to the existence of a state Φ on $\mathbb{B}(\mathcal{H})$ satisfying $\Phi(\pi(g)T) = \Phi(T\pi(g))$ for all $g \in G$, $T \in \mathbb{B}(\mathcal{H})$. One can ask if a similar criterion holds for amenable correspondences. When M is a II_1 factor, we will show that this indeed is the case if we replace $\mathbb{B}(\mathcal{H})$ with the von Neumann algebra $\mathcal{N} = \mathbb{B}(\mathcal{H}) \cap \pi_{M^\circ}(M^\circ)'$. That is, we obtain the following characterization of amenable correspondences:

Theorem 2.2 (Compare with Theorem 2.1 in [13].). *Let \mathcal{H} be an N - M correspondence with N finite with normal faithful trace τ and M a II_1 factor. Let $P \subset N$ be a von Neumann subalgebra. Then the following are equivalent:*

- (1) *there exists a net (ξ_n) in $\mathcal{H} \otimes_M \bar{\mathcal{H}}$ such that $\langle x\xi_n, \xi_n \rangle \rightarrow \tau(x)$ for all $x \in N$ and $\|[u, \xi_n]\| \rightarrow 0$ for all $u \in \mathcal{U}(P)$;*
- (2) *there exists a P -central state Φ on \mathcal{N} such that Φ is normal when restricted to N and faithful when restricted to $\mathcal{Z}(P' \cap N)$;*
- (3) *there exists a P -central state Φ on \mathcal{N} which restricts to τ on N .*

We do so by constructing a normal, faithful, semi-finite (tracial) weight $\bar{\tau}$ on \mathcal{N} which canonically realizes $\mathcal{H} \otimes_M \bar{\mathcal{H}}$ as $\mathcal{L}^2(\mathcal{N}, \bar{\tau})$. An identical construction to the one we propose has already appeared in the work of Sauvageot [25] for an arbitrary factor M . However, we present an elementary approach in the II_1 case.

Before presenting the details, we pause here to illustrate how Theorem 2.2 generalizes (relative) amenability for II_1 factors. Let M be a II_1 factor and $\mathcal{H} = L^2M \otimes L^2\bar{M}$ be the coarse M - M correspondence. We then have explicitly that $\mathcal{M} = \mathbb{B}(\mathcal{H}) \cap (M^\circ)' = \mathbb{B}(L^2M) \bar{\otimes} M$; hence, \mathcal{M} has an M -central state which restricts to the trace on M if and only if the trace on M extends to a hypertrace on $\mathbb{B}(L^2M)$, i.e., M is amenable. Similarly, if $P, Q \subset M$ are von Neumann subalgebras and $\mathcal{H} = L^2\langle M, e_Q \rangle$, where $\langle M, e_Q \rangle$ denotes the basic construction of

Jones, then $\langle M, e_Q \rangle \subset \mathbb{B}(\mathcal{H}) \cap (M^o)'$. So, if $P \subset M$ satisfies one of the conditions in the above theorem, then $P \triangleleft_M Q$ in the sense of Theorem 2.1 in [13]. Conversely, if $P \triangleleft_M Q$, then using condition (4) in Theorem 2.1 of [13], one can construct such a functional Φ on $\mathbb{B}(\mathcal{H}) \cap (M^o)'$ as in condition (3) in the above theorem for $P \subset M$.

Recall that a vector $\xi \in \mathcal{H}$ is *right bounded* if there exists $C > 0$ such that for all $x \in M$, $\|\xi x\| \leq C\|x\|_2$. The right-bounded vectors form a dense subspace of \mathcal{H} which we will denote by \mathcal{H}_b . Regarding \mathcal{H} as a right Hilbert M -module, we can define a natural M -valued inner product on \mathcal{H}_b , which we will denote $(\xi|\eta) \in M$ for $\xi, \eta \in \mathcal{H}_b$, by setting $(\xi|\eta)$ to be the Radon-Nikodym derivative of the normal functional $x \mapsto \langle \xi x, \eta \rangle$. Then it is easy to see that $(\cdot|\cdot)$ satisfies the following properties for all $\xi, \eta \in \mathcal{H}_b$, $x, y \in M$:

- (1) $(\xi|\xi) \geq 0$,
- (2) $(\eta|\xi) = (\xi|\eta)^*$,
- (3) $(\xi x|\eta y) = y^*(\xi|\eta)x$

(i.e., $(\mathcal{H}_b, (\cdot|\cdot))$ is an M -rigged space in the sense of Rieffel [24]). Trivially, we have that $\langle \xi, \eta \rangle = \tau((\xi|\eta))$. Also, by the non-degeneracy of π_{M^o} , $(\xi|\xi) = 0$ only if $\xi = 0$. Moreover, $(\cdot|\cdot)$ satisfies a noncommutative Cauchy-Schwartz inequality:

- (4) $(\eta|\xi)(\xi|\eta) \leq \|(\xi|\xi)\|_\infty(\eta|\eta)$

(cf. [23], Proposition 2.9).

Let $\mathcal{N} = \mathbb{B}(\mathcal{H}) \cap \pi_{M^o}(M^o)'$. (For instance, if $\tilde{M} \supset M$ is a tracial inclusion of II_1 factors and $\mathcal{H} = L^2\tilde{M}$, considered as a Hilbert M - M bimodule in the natural way, then we have that $(\hat{x}|\hat{y}) = E_M(y^*x)$ for all $x, y \in \tilde{M}$ and $\mathbb{B}(\mathcal{H}) \cap (M^o)' = \langle \tilde{M}, e_M \rangle$.) For $\xi, \eta \in \mathcal{H}_b$, let $T_{\xi, \eta} : \mathcal{H}_b \rightarrow \mathcal{H}_b$ be the ‘‘rank-one operator’’ given by $T_{\xi, \eta}(\cdot) = \xi(\cdot|\eta)$. Then $T_{\xi, \eta}$ extends to a bounded operator with $\|T_{\xi, \eta}\|_\infty^2 \leq \|(\xi|\xi)\|_\infty\|(\eta|\eta)\|_\infty$ [23]. Notice that $T_{\xi, \xi} \geq 0$ and that $T_{\xi, \xi}$ is a projection if $(\xi|\xi) \in \mathcal{P}(M)$. Since $T_{\xi, \eta}\pi_{M^o}(x) = \pi_{M^o}(x)T_{\xi, \eta}$ for all $x \in M^o$, we have that $T_{\xi, \eta} \in \mathcal{N}$. It is easy to see that the span of all such operators $T_{\xi, \eta}$ is a $*$ -subalgebra of $\mathbb{B}(\mathcal{H})$ which we will denote by \mathcal{N}_f . Noticing that for any $S \in \mathcal{N}$, $S(\mathcal{H}_b) \subset \mathcal{H}_b$, we have that $ST_{\xi, \eta} = T_{S\xi, \eta}$ and $T_{\xi, \eta}S = T_{\xi, S^*\eta}$. It follows that \mathcal{N}_f is an ideal of \mathcal{N} which can be considered as the analog of the finite-rank operators in $\mathbb{B}(\mathcal{H})$. The following lemma further cements this analogy.

Lemma 2.3. *If M is a II_1 factor, we have that $\mathcal{N}'_f \cap \mathbb{B}(\mathcal{H}) = \pi_{M^o}(M^o)$; hence, $\mathcal{N}_f + \mathbb{C}1_{\mathbb{B}(\mathcal{H})}$ is weakly dense in \mathcal{N} .*

Proof. The inclusion $\pi_{M^o}(M^o) \subset \mathcal{N}'_f \cap \mathbb{B}(\mathcal{H})$ is trivial. Conversely, let $T \in \mathcal{N}'_f \cap \mathbb{B}(\mathcal{H})$ and choose a non-zero $\zeta \in \mathcal{H}_b$ such that $(\zeta|\zeta) = p \in \mathcal{P}(M)$. (One can always find such a ζ as \mathcal{H} has an orthonormal basis of right bounded vectors as a right Hilbert M -module.) Choose a sequence $\eta_i \in \mathcal{H}_b$ such that $\|\eta_i - T\zeta\| \rightarrow 0$ and let $y_i = (\eta_i|\zeta)$. Then for every $\xi \in \mathcal{H}_b$ we have that $\|T(\xi p) - \xi y_i\| \leq \|T_{\xi, \zeta}\|_\infty\|\eta_i - T\zeta\| \leq \|(\xi|\xi)\|_\infty^{1/2}\|\eta_i - T\zeta\|$ so, the sequence $(\pi_{M^o}(y_i^o))$ converges in the strong topology to $T \circ \pi_{M^o}(p^o)$. Hence, $T \circ \pi_{M^o}(p^o) \in \pi_{M^o}(M^o)'' = \pi_{M^o}(M^o)$. Since M is a II_1 factor, by repeating the argument with $\zeta' = \zeta u$ for $u \in \mathcal{U}(M)$ and using standard averaging techniques, we conclude that there exists $y_T \in M$ such that $T\xi = \xi y_T$ for all $\xi \in \mathcal{H}$. Thus $T = \pi_{M^o}(y_T^o)$. \square

Now, consider an element $\varphi \in M_*$, and define a functional $\bar{\varphi} \in (\mathcal{N}_f)^*$ by $\bar{\varphi}(T_{\xi,\eta}) = \varphi((\xi|\eta))$. It is easy to see that $\bar{\varphi}$ is normal on \mathcal{N}_f and so, by the preceding lemma, may be extended to a normal semi-finite weight on \mathcal{N} . Hence, we may construct for each such φ a noncommutative L^p -space over \mathcal{N} , $\mathcal{L}^p(\mathcal{N}, \bar{\varphi}) = \{T \in \mathcal{N} : \|T\|_p = \bar{\varphi}(|T|^p)^{1/p} < \infty\}$. If M is a II_1 factor with trace τ , then $\bar{\tau}$ is a normal, faithful, semi-finite trace on \mathcal{N} and we denote $\mathcal{L}^p(\mathcal{N}, \bar{\tau})$ simply by $\mathcal{L}^p(\mathcal{N})$. In the case of $\mathcal{L}^2(\mathcal{N})$, we compute that $\|T_{\xi,\eta}\|_2^2 = \tau((\xi|\xi)(\eta|\eta)) = \langle \xi(\eta|\eta), \xi \rangle$. This shows that the map which sends $T_{\xi,\eta}$ to the elementary M -tensor $\xi \otimes_M \bar{\eta} \in \mathcal{H} \otimes_M \bar{\mathcal{H}}$ extends to an N - N bimodular Hilbert space isometry from $\mathcal{L}^2(\mathcal{N})$ to $\mathcal{H} \otimes_M \bar{\mathcal{H}}$. We are now ready to prove the motivating result in this section.

Proof of Theorem 2.2. The proof of (1) \Leftrightarrow (3) follows the usual strategy. For (1) \Rightarrow (3), we have that there exists a net (ξ_n) of vectors in $\mathcal{H} \otimes_M \bar{\mathcal{H}}$ such that $\langle x\xi_n, \xi_n \rangle \rightarrow \tau(x)$ for all $x \in N$ and $\|[u, \xi_n]\| \rightarrow 0$ for all $u \in \mathcal{U}(P)$. Viewing ξ_n as an element of $\mathcal{L}^2(\mathcal{N})$, let $\Phi_n \in \mathcal{N}_*$ be given by $\Phi_n(T) = \bar{\tau}(\xi_n \xi_n^* T)$ for any $T \in \mathcal{N}$. Then, by the generalized Powers-Størmer inequality (Theorem IX.1.2 in [28]), we have that $|\Phi_n(x) - \tau(x)| \rightarrow 0$ for all $x \in N$ and $\|\text{Ad}(u)\Phi_n - \Phi_n\|_1 \rightarrow 0$ for all $u \in \mathcal{U}(P)$. Taking a weak cluster point of (Φ_n) in \mathcal{N}^* gives the required N -tracial P -central state on \mathcal{N} . Conversely, given such a state Φ , we can find a net (η_n) in $\mathcal{L}^1(\mathcal{N})_+$ such that $\Phi_n(T) = \bar{\tau}(\eta_n T)$ weakly converges to Φ . In fact, by passing to convex combinations we may assume $\|[u, \eta_n]\|_1 \rightarrow 0$ for all $u \in \mathcal{U}(P)$. By another application of the generalized Powers-Størmer inequality, it is easy to check that $\xi_n = \eta_n^{1/2} \in \mathcal{L}^2(\mathcal{N}) \cong \mathcal{H} \otimes_M \bar{\mathcal{H}}$ satisfies the requirements of (1).

We now need only show (2) \Rightarrow (3) as (3) \Rightarrow (2) is trivial. But this is exactly the averaging trick found in the proof of (2) \Rightarrow (1) in Theorem 2.1 of [13]. We repeat the argument here for the sake of completeness. Since Φ is normal on N , we have that for some $\eta \in \mathcal{L}^1(N)_+$, $\Phi(x) = \tau(\eta x)$ for all $x \in N$. In fact, $\eta \in \mathcal{L}^1(P' \cap N)_+$ since Φ is P -central. Denoting by \mathcal{F} the net of finite subsets of $\mathcal{U}(P' \cap N)$ under inclusion, for any $\varepsilon > 0$, we set $\eta_F = \frac{1}{|F|} \sum_{u \in F} u \eta u^*$ and $\xi_{F,\varepsilon} = (\chi_{[\varepsilon, \infty)}(\eta_F)) \eta_F^{-1/2}$. We now let $\Psi_{F,\varepsilon}(T) = \frac{1}{|F|} \sum_{u \in F} \Phi(u \xi_{F,\varepsilon} T \xi_{F,\varepsilon} u^*)$. Note that $\Psi_{F,\varepsilon}$ is still P -central. Now it is easy to see that $\lim_{\mathcal{F}, \varepsilon} \chi_{[\varepsilon, \infty)}(\eta_F) = z$, where z is the central support of η . But by the faithfulness of Φ on $\mathcal{Z}(P' \cap N)$, we see that $z = 1$. Hence, any weak cluster point of $(\Psi_{F,\varepsilon})_{\mathcal{F}, \varepsilon}$ in \mathcal{N}^* is a P -central state which when restricted to N is τ . \square

Corollary 2.4 (generalized Haagerup's criterion for amenability). *Let N, M be II_1 factors and \mathcal{H} an N - M correspondence. If $P \subset N$ is a von Neumann subalgebra, then \mathcal{H} is left amenable over P (in the sense of Theorem 2.2) if and only if for every non-zero projection $p \in \mathcal{Z}(P' \cap N)$ and finite subset $F \subset \mathcal{U}(P)$, we have*

$$\left\| \sum_{u \in F} up \otimes \bar{u} p \right\|_{\mathcal{H} \otimes_M \bar{\mathcal{H}}, \infty} = |F|,$$

where $\|\cdot\|_{\mathcal{H} \otimes_M \bar{\mathcal{H}}, \infty}$ denotes the operator norm on $\mathbb{B}(\mathcal{H} \otimes_M \bar{\mathcal{H}})$.

Proof. Since we have obtained a ‘‘hypertrace’’ characterization of amenability for correspondences in Theorem 2.2, the result follows by the same arguments as in Lemma 2.2 in [7]. \square

Definition 2.5 (cf. Definition 1.3 in [18]). Let M be a II_1 factor, \mathcal{H} an M - M correspondence and P^c the orthogonal projection onto the subspace $\mathcal{H}^c = \{\xi \in \mathcal{H} : x\xi = \xi x, \forall x \in M\}$. The correspondence \mathcal{H} has *spectral gap* if for every $\varepsilon > 0$ there exist $\delta > 0$ and $x_1, \dots, x_n \in M$ such that if $\|x_i\xi - \xi x_i\| < \delta, i = 1, \dots, n$, then $\|\xi - P^c\xi\| < \varepsilon$. The correspondence \mathcal{H} has *stable spectral gap* if $\mathcal{H} \otimes_M \bar{\mathcal{H}}$ has spectral gap.

Note that if \mathcal{H} has stable spectral gap, then \mathcal{H} is amenable if and only if $(\mathcal{H} \otimes_M \bar{\mathcal{H}})^c \neq \{0\}$. Hence, we say an M - M correspondence \mathcal{H} is *nonamenable* if it has stable spectral gap and $(\mathcal{H} \otimes_M \bar{\mathcal{H}})^c = \{0\}$. The following theorem is the analog of Lemma 3.2 in [21] for the category of correspondences. N.B. Stable spectral gap as defined in [21] corresponds to our definition of nonamenability.

Theorem 2.6. *Let M be a II_1 factor and \mathcal{H} an M - M correspondence. Then \mathcal{H} is nonamenable if and only if $\mathcal{H} \otimes_M \bar{\mathcal{K}}$ has spectral gap and for any M - M correspondence \mathcal{K} .*

Proof. Let $\mathcal{H}_b, \mathcal{K}_b$ denote subspaces of right-bounded vectors in \mathcal{H} and \mathcal{K} , respectively. Given $\xi \in \mathcal{H}_b$ and $\eta \in \mathcal{K}_b$, by the same arguments as above we can define a bounded operator $T_{\xi, \eta} : \mathcal{K} \rightarrow \mathcal{H}$ by $T_{\xi, \eta}(\cdot) = \xi(\cdot|\eta)$. As above, one may check that $\|(T^*T)^{1/2}\|_2 = \|\xi \otimes_M \bar{\eta}\| = \|(TT^*)^{1/2}\|_2$ so that $\mathcal{H} \otimes_M \mathcal{K}$ is isometric to a Hilbert-normed subspace of the bounded right M -linear operators from \mathcal{H} to \mathcal{K} , which we denote $\mathcal{L}^2(\mathcal{H}, \mathcal{K})$. Moreover, this identification is natural with respect to the M - M bimodular structure on $\mathcal{L}^2(\mathcal{H}, \mathcal{K})$ given by $xT_{\xi, \eta}y = T_{x\xi, y^*\eta}$.

We need now only prove the forward implication, as the converse is trivial. Let us fix some arbitrary M - M correspondence \mathcal{K} . From Proposition 1.4 in [18], we have that if $(\mathcal{H} \otimes_M \bar{\mathcal{H}})^c = \{0\}$, then $(\mathcal{H} \otimes_M \bar{\mathcal{K}})^c = \{0\}$. So, by way of contradiction, we may assume that for every $\varepsilon > 0$ and $x_1, \dots, x_n \in M$, there exists a unit vector $\xi \in \mathcal{H} \otimes_M \bar{\mathcal{K}}$ such that $\|x_i\xi - \xi x_i\|_2 \leq \varepsilon, i = 1, \dots, n$. Without loss of generality, we may assume x_1, \dots, x_n are unitaries. Viewing ξ as an element of $\mathcal{L}^2(\mathcal{H}, \mathcal{K})$, let $\eta = (\xi^*\xi)^{1/2} \in \mathcal{L}^2(\mathcal{H})$. By the generalized Powers-Størmer inequality, we have $\|x_i\eta x_i^* - \eta\|_2^2 \leq 2\|x_i\xi x_i^* - \xi\|_2 \leq 2\varepsilon, i = 1, \dots, n$. Hence, $\eta \in \mathcal{L}^2(\mathcal{H}) \cong \mathcal{H} \otimes_M \bar{\mathcal{H}}$ is a unit vector such that $\|x_i\eta - \eta x_i\|_2 \leq \sqrt{2\varepsilon}, i = 1, \dots, n$. Thus, $\mathcal{H} \otimes_M \bar{\mathcal{H}}$ does not have spectral gap, a contradiction. \square

3. PROOFS OF MAIN THEOREMS

In this section we prove our main result, from which will follow Theorems 0.2 and 0.3. To begin, let Γ be an ICC countable discrete group which admits an unbounded cocycle $b : \Gamma \rightarrow \mathcal{K}$ for some orthogonal representation $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{K})$. Let $\Gamma \curvearrowright^\sigma (X, \mu)$ be the Gaussian construction associated to π as described in section 1.2 and $\{\omega_t : t \in \mathbb{R}\}$ be the one-parameter family of cocycles associated to b as given by Theorem 1.4. Let α_t be the $*$ -automorphism of $\tilde{M} = L^\infty(X, \mu) \rtimes \Gamma$ defined by $\alpha_t(au_\gamma) = a\omega_t(\gamma)u_\gamma$ for all $a \in L^\infty(X, \mu), \gamma \in \Gamma$. Finally, we set $M = L\Gamma$, and we denote by \mathcal{H} the M - M bimodule $L_0^2(X, \mu) \otimes \ell^2\Gamma$ with the usual bimodule structure; i.e., the one defined by $u_\gamma(\xi \otimes \eta) = \sigma(\gamma)\xi \otimes u_\gamma\eta, (\xi \otimes \eta)u_\gamma = \xi \otimes (\eta u_\gamma)$ for all $\xi \in L_0^2(X, \mu), \eta \in \ell^2\Gamma$ and $\gamma \in \Gamma$.

Theorem 3.1. *With the assumptions and notations as above, suppose $P \subset M$ is a diffuse von Neumann subalgebra such that $\mathcal{N}_M(P)$ acts weakly compactly on P via conjugation. Let $Q = \mathcal{N}_M(P)''$. If either: (1) b is a proper cocycle; or (2) π is a mixing representation and α_t does not converge $\|\cdot\|_2$ -uniformly to the identity on $(Qp)_1$ for any projection $p \in \mathcal{Z}(Q' \cap M)$ as $t \rightarrow 0$, then the M - M correspondence \mathcal{H} is left amenable over Q in the sense of satisfying Theorem 2.2 for $Q \subset M$.*

Proof. In the case of (1), since b is proper, it is easy to see by formula 1.2 that $E_{L\Gamma} \circ \alpha_t$ restricted to $L\Gamma$ is compact for all $t > 0$. Hence, by the proof of Theorem 4.9 in [13], for any $K \geq 8$, any non-zero projection $p \in \mathcal{Z}(Q' \cap M)$, and any finite subset $F \subset \mathcal{N}_M(P)$, we can find a vector $\xi_{p,F} \in \mathcal{H} \otimes L^2(\tilde{M})$ such that $\|x\xi_{p,F}\| \leq \|x\|_2$ for all $x \in M$, $\|p\xi_{p,F}\| \geq \|p\|_2/K$ and $\|[u \otimes \bar{u}, \xi_{p,F}]\| < 1/|F|$ for all $u \in F$.

In the case of (2), we need only demonstrate that our assumptions imply the existence of such a net $(\xi_{p,F})$ as in case (1) for some $K \geq 8$ then argue commonly for both sets of assumptions. By contradiction if such a net $(\xi_{p,F})$ did not exist for any $K \geq 8$, the proof of Theorem 4.9 in [13] shows that for every $0 < \delta = K^{-1} \leq 1/8$ and for any $t > 0$ sufficiently small we have

$$(3.1) \quad \|E_M \circ \alpha_t(up)\|_2 \geq (1 - 6\delta)\|p\|_2$$

for all $u \in \mathcal{U}(P)$. Now, the operators $(E_M \circ \alpha_t)_{t \geq 0}$ can be seen to form a one-parameter semigroup of unital, tracial completely-positive maps (cf. Example 2.2 in [17]) such that $0 \leq E_M \circ \alpha_t \leq \text{id}$ for all $0 \leq t < \infty$. This implies that

$$(3.2) \quad \|x\|_2^2 - \|E_M \circ \alpha_t(x)\|_2^2 \geq \|x - E_M \circ \alpha_t(x)\|_2^2.$$

Hence, if α_t does not converge uniformly on $(Pp)_1$, we have that α_t cannot converge uniformly on $\mathcal{U}(P)p$, so there exists $c > 0$ such that for every $t > 0$ sufficiently small, there exists $u_t \in \mathcal{U}(P)$ such that

$$\|E_M \circ \alpha_t(u_t p)\|_2 \leq \sqrt{1 - c^2}\|p\|_2.$$

However, this contradicts the inequality 3.1 for δ sufficiently small.

To conclude the discussion of case (2), we have that π is mixing and, by the previous paragraph, α_t converges $\|\cdot\|_2$ -uniformly on $(Pp)_1$. We will show that α_t converges uniformly on $(Qp)_1$, which contradicts our assumptions on α_t . Since there is a natural trace-preserving automorphism $\beta \in \text{Aut}(\tilde{M})$ which pointwise fixes M and such that $\beta \circ \alpha_t = \alpha_{-t} \circ \beta$ for all $t \in \mathbb{R}$ (cf. [18]), by Popa's transversality lemma [21], it is enough to show that $\|\alpha_t(x) - E_M \circ \alpha_t(x)\|_2 \rightarrow 0$ uniformly on $(Qp)_1$. Notice that $\delta_t(x) = \alpha_t(x) - E_M \circ \alpha_t(x) = (1 - E_M)(\alpha_t(x) - x)$ is a (bounded) derivation $\delta_t : M \rightarrow \mathcal{H}$. Since π is mixing, by [16] we have that \mathcal{H} is a compact correspondence; hence, by Theorem 4.5 in [17] $\delta_t \rightarrow 0$ uniformly in $\|\cdot\|_2$ -norm on $(Qp)_1$ as $t \rightarrow 0$.

Now, fixing a suitable $K \geq 8$ and proceeding commonly for both cases, let \mathcal{F} be the net of finite subsets of $\mathcal{N}_M(P)$ under inclusion. We define the state Φ_p on $\mathcal{N} = \mathbb{B}(\mathcal{H}) \cap \pi_{M^\circ}(M^\circ)'$ by

$$\Phi_p(T) = \text{Lim}_{\mathcal{F}} \frac{1}{\|p\xi_{p,F}\|_2^2} \langle (T \otimes 1)p\xi_{p,F}, p\xi_{p,F} \rangle,$$

where $\text{Lim}_{\mathcal{F}}$ is an arbitrary Banach limit. It is easy to see by the properties of $\xi_{p,F}$ that Φ_p is normal on M . Proceeding as in Lemma 5.3 in [14], we then have that for all $u \in \mathcal{N}_M(P)$,

$$\begin{aligned}
(3.3) \quad \Phi_p(u^*Tu) &= \text{Lim}_{\mathcal{F}} \frac{1}{\|p\xi_{p,F}\|_2^2} \langle (T \otimes 1)up\xi_{p,F}, up\xi_{p,F} \rangle \\
&= \text{Lim}_{\mathcal{F}} \frac{1}{\|p\xi_{p,F}\|_2^2} \langle (T \otimes 1)p(u \otimes \bar{u})\xi_{p,F}, p(u \otimes \bar{u})\xi_{p,F} \rangle \\
&= \text{Lim}_{\mathcal{F}} \frac{1}{\|p\xi_{p,F}\|_2^2} \langle (T \otimes 1)p(u \otimes \bar{u})\xi_{p,F}(u \otimes \bar{u})^*, p(u \otimes \bar{u})\xi_{p,F}(u \otimes \bar{u})^* \rangle \\
&= \Phi_p(T).
\end{aligned}$$

Hence, we have that $\Phi_p([x, T]) = 0$ for all x in the span of $\mathcal{N}_M(P)$ and $T \in \mathcal{N}$. But we have that

$$\begin{aligned}
(3.4) \quad |\Phi_p(Tx)| &\leq \|T\|_{\infty} \text{Lim}_{\mathcal{F}} \frac{1}{\|p\xi_{p,F}\|_2^2} \langle xp\xi_{p,F}, p\xi_{p,F} \rangle \\
&\leq \|T\|_{\infty} \text{Lim}_{\mathcal{F}} \frac{1}{\|p\xi_{p,F}\|_2} \|xp\xi_{p,F}\|_2 \leq \frac{K}{\|p\|_2} \|T\|_{\infty} \|x\|_2
\end{aligned}$$

and similarly for $|\Phi_p(xT)|$. Thus, by Kaplansky's density theorem we have that Φ_p is a Q -central state.

To summarize, for every non-zero projection $p \in \mathcal{Z}(Q' \cap M)$, we have obtained a state Φ_p on \mathcal{N} such that $\Phi_p(p) = 1$, Φ_p is normal on M and Φ_p is Q -central. A simple maximality argument then shows that there exists a state Φ on \mathcal{N} which is normal on M , Q -central, and faithful on $\mathcal{Z}(Q' \cap M)$. Thus, Φ satisfies condition (2) of Theorem 2.2, and we are done. \square

Keeping with the same notations, assume now that the orthogonal representation $b : \Gamma \rightarrow \mathcal{O}(\mathcal{K})$ is such that there exists an $K > 0$ such that $\pi^{\otimes K}$ is weakly contained in the left regular representation. As was pointed out in section 1.2, the representation induced on $L_0^2(X, \mu)$ by $\Gamma \curvearrowright^{\sigma} (X, \mu)$ also has this property. Let $\mathcal{H}_{\sigma} = L_0^2(X, \mu)$ so that $\mathcal{H} = \mathcal{H}_{\sigma} \otimes \ell^2\Gamma$ is the M - M correspondence induced by the representation σ . Denote by $\tilde{\mathcal{H}}^n$ the M - M correspondence $((\mathcal{H}_{\sigma} \otimes \tilde{\mathcal{H}}_{\sigma})^{\otimes n}) \otimes \ell^2\Gamma$ with the natural bimodule structure. It is straightforward to check that $\mathcal{H} \otimes_M \tilde{\mathcal{H}} \cong \mathcal{H}_{\sigma} \otimes \ell^2\Gamma \otimes \tilde{\mathcal{H}}_{\sigma}$ and that $(\mathcal{H} \otimes_M \tilde{\mathcal{H}}) \otimes_M \cdots \otimes_M (\mathcal{H} \otimes_M \tilde{\mathcal{H}})$ for $n + 1$ copies is isomorphic to $\mathcal{H} \otimes_M (\tilde{\mathcal{H}}^n) \otimes_M \tilde{\mathcal{H}}$ as M - M bimodules. Hence, the M -tensor product of K copies of $\mathcal{H} \otimes_M \tilde{\mathcal{H}}$ is weakly contained in the coarse M - M correspondence.

Theorem 3.2. *With the assumptions and notations as above, including those assumed for Theorem 3.1, suppose $P \subset M$ is a diffuse von Neumann subalgebra such that $\mathcal{N}_M(P)$ acts weakly compactly on P via conjugation. Then $Q = \mathcal{N}_M(P)''$ is amenable.*

Proof. Let p be a non-zero projection in $\mathcal{Z}(Q' \cap M)$. By the proofs of Theorems 3.1 and 2.2, it follows that we can find a net (ξ_n) in $\mathcal{H} \otimes_M \tilde{\mathcal{H}}$ such that $\langle x\xi_n, \xi_n \rangle \rightarrow \tau(pxp)/\tau(p)$ for all $x \in M$ and $\|[u, \xi_n]\| \rightarrow 0$ for all $u \in \mathcal{U}(Q)$. In fact, without loss of generality we may assume that $\langle x\xi_n, \xi_n \rangle = \tau(pxp)/\tau(p) = \langle \xi_n x, \xi_n \rangle$ for all $x \in M$ (see the proof of Proposition 3.2 in [13]). In particular, (ξ_n) is uniformly left and right bounded. Let $\tilde{\xi}_n$ be the M -tensor product

of K copies of ξ_n . Then the net $(\tilde{\xi}_n)$ may be seen to satisfy the same properties. Since $(\tilde{\xi}_n)$ are vectors in a correspondence weakly contained in the coarse M - M correspondence, we have that for any finite subset $F \subset \mathcal{U}(Q)$ that

$$\left\| \sum_{u \in F} up \otimes \overline{up} \right\|_{M \otimes \bar{M}} \geq \lim_n \left\| \sum_{u \in F} u \tilde{\xi}_n u^* \right\| = |F|.$$

Hence, by Haagerup's criterion [7], Q is amenable. \square

Remark 3.3. Let M be a II_1 factor and δ a closable real derivation from M into an M - M correspondence \mathcal{H} (cf. [17]). Suppose $P \subset M$ is a von Neumann subalgebra and $\mathcal{N}_M(P)$ acts weakly compactly on P by conjugation. Let $Q = \mathcal{N}_M(P)''$. One can show that if $\delta^* \bar{\delta}$ has compact resolvents, then \mathcal{H} is left amenable over Q . The proof is identical to the proof of Theorem B in [14], using the generalized Haagerup's criterion (Corollary 2.4). In particular, this sharpens Theorem A in [14]: any II_1 factor M with the CMAP admitting such a derivation into a nonamenable correspondence has no Cartan subalgebras.

We are now ready to prove Theorems 0.2 and 0.3.

Proof of Theorem 0.2. We need only check the cases $\text{SO}(n, 1)$, $n \geq 4$, and $\text{SU}(m, 1)$, $m \geq 2$, as $\text{SO}(1, 1)$ is amenable and the remaining cases were dealt with by Ozawa–Popa [14]. If Γ is an ICC lattice in $\text{SO}(n, 1)$ for $n \geq 3$ or $\text{SU}(m, 1)$ for $m \geq 2$, then Theorems 1.9 in [27] shows that Γ possesses an unbounded cocycle into some strongly ℓ^p representation for $p \geq 2$. By Theorem 3.4 in the same, any unbounded cocycle for such a lattice is proper. Since $L\Gamma$ has the CMAP by [4] and [8], by Theorems 1.7 and 3.2 the result obtains. \square

Proof of Theorem 0.3. For the first assertion, since $\Gamma \cong \Gamma_1 * \Gamma_2$, Γ admits a canonical unbounded cocycle $b : \Gamma \rightarrow \bigoplus_{\infty} \ell^2 \Gamma$ into a direct sum of left regular representations. The left regular representation is mixing and $L\Gamma$ - $L\Gamma$ correspondence associated to the left regular representation (the coarse correspondence) is amenable if and only if Γ is amenable. So, if $L\Gamma$ did admit a Cartan subalgebra, then by Theorems 1.7 and 3.1 the deformation α_t of $L\Gamma$ obtained from the cocycle b would have to converge uniformly on $(L\Gamma)_1$ as $t \rightarrow 0$. But this contradicts that b is unbounded.

For the second assertion, if a non-amenable II_1 subfactor $N \subset L\Gamma$ admits a Cartan subalgebra, then we have that α_t converges $\|\cdot\|_2$ -uniformly on the unit ball of N , since N has the CMAP and the coarse $L\Gamma$ - $L\Gamma$ correspondence viewed as an N - N correspondence embeds into a direct sum of coarse N - N correspondences. Let $\tilde{\Gamma} = \Gamma * \mathbb{F}_2$, where \mathbb{F}_2 is the free group on two generators. Let $u_1, u_2 \in L\mathbb{F}_2$ be the canonical generating unitaries and $h_1, h_2 \in L\mathbb{F}_2$ self-adjoint elements such that $u_j = \exp(\pi i h_j)$, $j = 1, 2$. Define $u_j^t = \exp(\pi i t h_j)$, $j = 1, 2$, and let θ_t be the $*$ -automorphism of $L\tilde{\Gamma}$ given by $\theta_t = \text{Ad}(u_1^t) * \text{Ad}(u_2^t)$. It follows from Lemma 5.1 in [17] and Corollary 4.2 in [18] that θ_t converges uniformly in $\|\cdot\|_2$ -norm on the unit ball of $N \subset L\Gamma \subset L\tilde{\Gamma}$ as $t \rightarrow 0$. An examination of the proof of Theorem 4.3 in [11] shows that this is the only condition necessary for the theorem to obtain. Our result then follows directly from Theorem 5.1 in [11]. \square

ACKNOWLEDGEMENTS

The author is grateful to Jesse Peterson for suggesting the problem and making several helpful comments and suggestions. The author also thanks Jesse Peterson and Ionut Chifan for many stimulating discussions on the work of Ozawa and Popa and the referee for catching several errors in the original manuscript as well as for suggesting several improvements in the exposition.

REFERENCES

1. C. Anantharaman-Delaroche, *Amenable correspondences and approximation properties for von Neumann algebras*, Pacific J. Math. **171** (1995), no. 2, 309–341.
2. B. Bekka, *Amenable representations of locally compact groups*, Invent. Math. **100** (1990), no. 2, 383–401.
3. N. Brown and N. Ozawa, *C*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, AMS, 2008.
4. J. de Cannière and U. Haagerup, *Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups*, Amer. J. Math. **107** (1985), 455–500.
5. M. Cowling and U. Haagerup, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one*, Invent. Math. **96** (1989), no. 3, 507–549.
6. M. Cowling, U. Haagerup, and R. Howe, *Almost L^2 matrix coefficients*, J. Reine Angew. Math. **387** (1988), 97–110.
7. U. Haagerup, *Injectivity and decomposition of completely bounded maps*, Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983), Lecture Notes in Math., no. 1132, Springer, Berlin, 1985, pp. 170–222.
8. ———, *Group C*-algebras without the completely bounded approximation property*, preprint, 1986.
9. C. Houdayer, *Strongly solid group factors which are not interpolated free group factors*, Math. Ann. **346** (2010), 969–989.
10. C. Houdayer and D. Shlyakhtenko, *Strongly solid II_1 factors with an exotic masa*, preprint, arXiv:0904.1225, 2009.
11. A. Ioana, J. Peterson, and S. Popa, *Amalgamated free products of weakly rigid factors and calculation of their symmetry groups*, Acta Math. **200** (2008), 85–153.
12. N. Ozawa, *Solid von Neumann algebras*, Acta Math. **192** (2004), 111–117.
13. N. Ozawa and S. Popa, *On a class of II_1 factors with at most one Cartan subalgebra*, Ann. of Math. (2), accepted, arXiv:0706.3623.
14. ———, *On a class of II_1 factors with at most one Cartan subalgebra, II*, Amer. J. Math. **132** (2010), 841–866.
15. K. R. Parthasarathy and K. Schmidt, *Infinitely divisible projective representations, cocycles and the Levi-Khinchine formula on locally compact groups*, preprint, 1970.
16. J. Peterson, *Examples of group actions which are virtually W^* -superrigid*, preprint, arXiv:1002.1745, 2009.
17. ———, *L^2 -rigidity in von Neumann algebras*, Invent. Math. **175** (2009), 417–33.
18. J. Peterson and T. Sinclair, *On cocycle superrigidity for Gaussian actions*, preprint, arXiv:0910.3958, 2009.
19. S. Popa, *Correspondences*, INCREST preprint, 1986.
20. ———, *On a class of type II_1 factors with Betti number invariants*, Ann. Math. (2) **163** (2006), no. 3, 809–99.
21. ———, *On the superrigidity of malleable actions with spectral gap*, J. Amer. Math. Soc. **21** (2008), 981–1000.

22. S. Popa and S. Vaes, *Group measure space decomposition of II_1 factors and W^* -superrigidity*, preprint, 2009, arXiv:math.OA/0906.2765.
23. M. A. Rieffel, *Induced representations of C^* -algebras*, *Advances in Math.* **13** (1974), 176–257.
24. ———, *Morita equivalence for C^* -algebras and W^* -algebras*, *J. Pure Appl. Algebra* **5** (1974), 51–96.
25. J.-L. Sauvageot, *Sur le produit tensoriel relatif d'espaces de Hilbert*, *J. Operator Theory* **9** (1983), no. 2, 237–252.
26. K. Schmidt, *From infinitely divisible representations to cohomological rigidity*, *Analysis, Geometry and Probability*, *Texts Read. Math.*, no. 10, Hindustan Book Agency, Dehli, 1996, pp. 173–97.
27. Y. Shalom, *Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group*, *Ann. of Math. (2)* **152** (2000), no. 1, 113–182.
28. M. Takesaki, *Theory of operator algebras II*, *Encyclopaedia of Mathematical Sciences*, no. 127, Springer, 2003.
29. D. Voiculescu, *The analogues of entropy and of Fischer's information measure in free probability theory. III. The absence of Cartan subalgebras*, *Geom. Funct. Anal.* **6** (1996), no. 1, 172–199.

THOMAS SINCLAIR, VANDERBILT UNIVERSITY, 1326 STEVENSON CENTER, NASHVILLE, TN 37240
E-mail address: thomas.sinclair@vanderbilt.edu