Math 274 Several Elementary Principles

1. Elementary Principles

Several basic counting principles:

- **Sum principle (counting by cases):** If a finite set A is partitioned into sets $B_1, B_2, \ldots, B_k$, then $|A| = \sum_{i=1}^{k} |B_i|$.
- **Product principle (counting by stages):** If the elements of A are build via (independent) successive choices, then $|A$ is the product of the numbers of the options for the successive choices.

  Example: There are $\prod_{i=0}^{n-1}(2n-1-2i)$ ways to pair $2n$ people. (hint: pair least index one with another one)
- **Principle of counting two ways:** when two formulas count the same set, their values are equal.

  Example: Prove that $\sum_{i=1}^{n-1} i = n(n-1)/2$. (hint: pick two members from $[n]$, and left side counts the pairs according to the larger one)
- **Bijection Principle:** If there is a bijection from one set to another, then the two sets have the same size.

  Examples: The number of $0,1$-lists of length n equals the number of subsets of $[n]$.
- Pigeonhole Principle (will talk more later)
- Polynomial Principle: if two polynomials in x are equal for infinitely many values of x, then they are the same polynomial.

2. Words, sets, and multisets

The difference between word and set is whether there is an order of the elements. Also, repetition may or may not allowed in words and sets. So we discuss four classical models.

(1) k**-word:** there are n^k k-words from an alphabet S of size n.

(2) **simple** k**-word:** there are $n_{(k)}$ simple k-words from an alphabet of size n.

(3) k**-sets:** there are $\binom{n}{k}$ k-sets in an n-set.

Remark: $\binom{n}{k}$ is called **binomial coefficients**. Here is the Binomial Theorem:

Binomial Theorem: For $n \in N$, $(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$.

Proof 1: direct counting. Collect terms.

Proof 2: Polynomial Principle: think $(x+y)^n$ to be n-words from an alphabet of size $x+y$.

(4) k**-multisets:** the number of k-element multisets from $[n]$ equals to the number of solutions to $\sum_{i=1}^{n} x_i = k$ in nonnegative integers, which equals to $\binom{k+n-1}{n-1} = \binom{k+n-1}{k}$.

This equation model is useful. Here is another example.

Example: What is the number of integral solutions of the equation $x_1+x_2+x_3+x_4 = 10$ in which $x_1 \geq 3, x_2 \geq 1, x_3 \geq 0$ and $x_4 \geq 5$?