

The Even Isomorphism Theorem for Coxeter Groups

M. Mihalik

July 16, 2003

1 Introduction.

A *Coxeter system* is a pair (W, S) such that W is a group with *Coxeter presentation* $\langle S : (st)^{m_{st}} \forall s, t \in S \rangle$ where for all $s, t \in S$, $m_{st} \in \{1, 2, \dots, \infty\}$, $m_{st} = m_{ts}$ and $m_{st} = 1$ if and only if $s = t$. (The relation $(st)^\infty$ means that st has infinite order in W .) Note that all $s \in S$ are order 2 and that if $m_{st} = 2$ then s and t commute.

There are two diagrams associated to a Coxeter group that appear regularly in the literature. The diagrams $V_D(W, S)$ and $V_F(W, S)$ for the Coxeter system (W, S) are labeled graphs with vertex set S . In V_F there is an edge labeled m_{st} between distinct vertices s and t if and only if $m_{st} \neq \infty$. In V_D there is a edge labeled m_{st} between distinct vertices s and t if and only if $m_{st} \neq 2$. The vertices of components of V_D generate factors of a direct product decomposition of W and the vertices of components of V_F generate factors of a free product decomposition of W . While it is traditional to call V_D the Coxeter graph or Coxeter diagram for (W, S) , in this paper we only consider V_F diagrams and we call such diagrams, Coxeter diagrams or simply diagrams.

A Coxeter presentation is *even* if all m_{st} for $s \neq t$ are even or ∞ . In this case we call the corresponding Coxeter group and diagram even.

A Coxeter group is *rigid* if any two Coxeter presentations for this group are isomorphic presentations. In [1], Bahls shows that any Coxeter group can have at most one even presentation. In § 4, we classify the even rigid Coxeter groups. Rigidity and a variety of analogous notions are designed to give insight into a fundamental problem in the theory of Coxeter groups.

The Coxeter Isomorphism Question: Given two Coxeter presentations, do they present isomorphic groups?

The Coxeter presentations $\langle x, y, z : x^2, y^2, z^2, (xy)^3, (xz)^2, (yz)^2 \rangle$ and $\langle a, b : a^2, b^2, (ab)^6 \rangle$ present isomorphic groups, but only the latter is even. In particular, the Coxeter group presented here is not rigid.

In this paper we produce an algorithm to decide if an arbitrary Coxeter presentation presents a finitely generated even Coxeter group. Furthermore, we can decide if two (finite) Coxeter presentations present the same even Coxeter group. This solves the even Coxeter isomorphism question.

Our Proposition 7 is used by Patrick Bahl's in his thesis [1] to show that there is an unique even Coxeter presentation for a finitely generated even Coxeter group. We in turn use Bahl's result in the final stage of our algorithm to decide if two finite Coxeter presentations present the same even Coxeter group. The proof of a vital combinatorial lemma hinges on the visual decomposition theorem of [6]. A critical tool in our algorithm is that of twisting in Coxeter diagrams. This method of producing different Coxeter diagrams (and different presentations) for the same Coxeter group was introduced by N. Brady, J. McCammond, B. Muhlherr and W. Neumann in [5]. At this time the only known way to produce different Coxeter diagrams for a given Coxeter group is by twisting or by a certain triangle/edge exchange.

Our main theorem is the following:

Theorem 1 *Suppose (W, S) is an even Coxeter system and V' is a Coxeter diagram for W with odd labeled edge $[xy]$. Then there is a diagram V'' for W obtained from V' by first performing a twist around $[xy]$ and then replacing a triangle $[xyu]$ by an edge with even label.*

The proof of our theorem specifically defines the set to be twisted and a vertex u so that triangle $[xyu]$ may be replaced by an even edge. The resulting diagram for W has (one) fewer odd labeled edges than the original. Thus we have a simple algorithm to change a non-even diagram for a finitely generated even Coxeter group W into the unique even diagram for W .

If (W, S) is an arbitrary Coxeter system with diagram V containing an odd edge $[xy]$, then either the described twist and triangle replacement can be carried out or W is not an even Coxeter group. Hence one can decide if a given finitely generated Coxeter group is even or not. Given two finitely generated Coxeter systems (W_1, S_1) and (W_2, S_2) with diagrams V_1 and V_2 respectively, one can decide if W_1 and W_2 are isomorphic even Coxeter groups. Simply

apply our algorithm repeatedly to V_1 and V_2 until either an odd edge cannot be replaced by an even one using our technique (in which case one of the groups is not even), or until all odd edges are replaced in both diagrams. In the later scenario, Bahls' even rigidity result implies that the resulting even diagrams are diagram isomorphic if and only if W_1 and W_2 are isomorphic.

It is also evident that given a Coxeter system for a finitely generated even Coxeter group, one can use the methods of this paper to produce all other Coxeter systems for that group.

2 Preliminaries.

In this section, we describe: twisting in Coxeter diagrams as introduced in [5], visual decompositions of Coxeter groups [6], and techniques to construct quotient maps of Coxeter groups that match quotient maps of Coxeter diagrams.

(1) Twisting. In an arbitrary Coxeter system (W, S) , twisting makes sense for any subset of S that generates a finite subgroup of W . We only need twist around pairs of distinct vertices $x, y \in S$ such that m_{xy} is an odd integer.

Suppose V is a Coxeter diagram for the Coxeter system (W, S) . Given $x, y \in S$, define $lk(x)$ (the “link” of x) to be the set of all vertices of V that are connected to x by an edge. Define $lk_2(x)$ (the “2-link” of x) to be the set of all vertices of V that are connected to x by an edge labeled 2. Define $st(x)$ (the “star” of x) to be $lk(x) \cup \{x\}$. Define $lk_2(x, y)$ (the “2-link” of x and y) to be $lk_2(x) \cap lk_2(y)$, i.e. the set of all vertices in V that are connected to both x and y by an edge labeled 2. So each $s \in lk_2(x, y)$ commutes with both x and y . Denote $b^{-1}ab$ by a^b . Now suppose x and y are distinct elements of S and $m_{xy} = 2n + 1$. Let d be the (unique) element of length $2n + 1$ in $\langle x, y \rangle$. Note that $x^d = y$ and $y^d = x$. Suppose $U \subset S - \{x, y\}$ and for each edge $[us]$ such that $s \in S - (U \cup \{x, y\})$ and $u \in U$, $s \in lk_2(x, y)$. Then the twisting theorem of [5] implies that (W, S') is a Coxeter system, where $S' = U^d \cup (S - U)$ and a diagram for (W, S') is obtained from V by changing each edge of V that connects to a vertex $u \in U$ to a vertex $v \in \{x, y\}$ to connect instead from u to v^d , and leaving other edges unchanged.

(2) Visual Decompositions of Coxeter Groups. Suppose V is a diagram for the Coxeter system (W, S) and some subset C of S separates vertices of V , then a simple examination of presentations, shows that W decomposes

as $\langle A \rangle *_{\langle C \rangle} B$, where $A \cup B = S$, A is C union the vertices of some set of components of $V - C$ and B is $C \cup (S - A)$. This type of decomposition extends in a natural way to graphs of groups decompositions of W . Such decompositions are called “visual” decompositions of W since they are easily seen in V and the main theorem of [6] states that given any graph of groups decomposition of W there is a visual decomposition that basically refines the given decomposition. More specifically, any vertex (edge) group of the visual decomposition of W is a subgroup of a conjugate of a vertex (edge) group of the given decomposition. For our purposes this result is particularly useful when we have two different diagrams for W so that visual decompositions with respect to the two diagrams can be played against one another.

(3) Coxeter Quotients. Suppose (W, S) is a Coxeter system with diagram V . If $T \subset W$ then let $N(T)$ be the normal closure of T in W . If $T \subset S$ then $W/N(T)$ is a Coxeter group with diagram obtained from V by removing the vertices of T and all vertices that connect to a vertex of T by a path with all odd labeled edges. In this paper, we often consider a diagram for an even Coxeter system (W, S) and another diagram V' for the system (W, S') where V' may have odd labeled edges. Our Proposition 7 describes a 1-1 correspondence between the set of edges with label > 2 in V and those edges with label > 2 in V' . If an edge $[xy]$ of V' has odd label and $[xy]$ corresponds to the edge $[ab]$ of V , then in fact, the cyclic group $\langle xy \rangle$ is conjugate to the group $\langle (ab)^2 \rangle$. A diagram for $W/N(xy)$ is obtained from V' by collapsing the edge $[xy]$ to a point. If $[xyu]$ is triangle then our Lemma 9 states that $[xu]$ and $[yu]$ are labeled 2. Hence additional “collapsing” in V' is not generated by the collapsing of $[xy]$ (see Lemma 10). A diagram for $W/N(xy)$ is obtained from V by changing the label on $[ab]$ to 2. In this way, we can be sure that $W/N(xy)$ is an even Coxeter group with a diagram that preserves potentially desirable aspects of V' . Other quotients of diagrams for (W, S) and (W, S') are obtained when we find subsets $\sigma \subset S$ and $\sigma' \subset S'$ such that $\langle \sigma \rangle$ and $\langle \sigma' \rangle$ are conjugate. If $f : \langle \sigma' \rangle \rightarrow \mathbb{Z}_2 (= \{-1, 1\})$ is a homomorphism, and N is the normal closure in W of $\ker(f)$, then Lemma 29 describes how to obtain an even diagram for W/N from V . Understanding how quotients of W correspond to quotients of two different diagrams for W is crucial to the success of our arguments in this paper.

3 A Reduction and Outline

The main theorem can be reduced to a combinatorial fact.

Theorem 2 *Suppose W is a finitely generated even Coxeter group and V' is a diagram for W . If l is a simple (does not cross itself) edge loop in V' of length ≥ 4 and containing an odd labeled edge, then there is an edge of V' containing two non-consecutive vertices of l . (I.e. there is a shortcut in l .)*

In Section 6, we prove Theorem 2 for loops of length 4 and in Section 7 we show all other cases reduce to the length 4 case, finishing the proof of Theorems 1 and 2. In this section we prove:

Proposition 3 *Suppose W is a finitely generated even Coxeter group and V' is a diagram for W with odd labeled edge $[xy]$ such that every simple closed edge loop containing $[xy]$ of length ≥ 4 has a shortcut, then after a twist, a triangle containing $[xy]$ can be replaced by an edge with even label. (In particular, the main theorem can be reduced to Theorem 2.)*

Proof: Our hypothesis and Lemma 9 (below) imply:

Lemma 4 *Every path in V' from x to y either contains the edge $[xy]$ or intersects $lk_2(x, y)$. •*

Suppose U is the union of all components C of $V' - (\{x, y\} \cup lk_2(x, y))$ such that there is an edge from x to C . Then by Lemma 4, there is no edge from y to U . If there is a vertex $t \in V' - (\{x, y\} \cup U)$ that connects to U by an edge, then $t \in lk_2(x, y)$ and so U can be twisted around $\{x, y\}$, to form the diagram \hat{V}' for W . Note that after twisting, vertices of U that were connected to x are replaced in \hat{V}' by vertices that connect to y instead. I.e. in \hat{V}' , each edge (other than $[xy]$) containing x has its other vertex in $lk_2(x, y)$.

Proposition 5 *Suppose V is an even diagram for the Coxeter group W , $[xy]$ is an odd edge in a diagram V' for W and every edge $[xc]$ of V' for $c \neq y$ is such that $c \in lk_2(x, y)$. Then there exists a vertex $u \in lk_2(x, y)$ such that if $[uc]$ is an edge with $c \notin \{x, y\}$, then $[uc]$ has label 2, $c \in lk_2(x, y)$, and any simplex σ' of V' that contains u (respectively x and y) and is such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for σ a simplex of V , contains $\{x, y\}$ (respectively u).*

This proposition implies that the triangle $[xyu]$ of V' can be replaced by the edge $[yb]$ with label 2 times the order of xy , finishing Proposition 3. More specifically, form the diagram \hat{V}' from V' by removing the vertices x and u , adding a vertex b and edge from y to b with label 2 times the order of xy , and for each vertex c of $lk(x) - \{u, y\} = lk(u) - \{x, y\}$, add an edge labeled 2 from c to b . Then \hat{V}' is a diagram for W satisfying the conclusion of Proposition 3. •

The proof of Proposition 5 requires the development of some basic results and is postponed until Section 5.

4 Classifying the Rigid Even Coxeter Groups

In this section we develop several important tool lemmas and prove Theorem 26, a classification of the rigid even Coxeter groups. Through the remainder of the paper we rely on [4] as a reference for basic facts about Coxeter groups.

If V is a diagram for a Coxeter group W , then a simplex σ is *spherical* if $\langle \sigma \rangle$ is a finite subgroup of W and σ is *maximal spherical* if σ is spherical and properly contained in no other spherical simplex. Maximal spherical simplices of V give (up to conjugation), the maximal finite subgroups of W . Hence if V and V' are diagrams for W and σ is a maximal spherical simplex of V , then there is a maximal spherical simplex σ' of V' such that $\langle \sigma \rangle$ is conjugate to $\langle \sigma' \rangle$.

For any Coxeter diagram V , let $T(V)$ be the product of all edge labels of V .

Remark 1 For any integer k , $D_{2(2k+1)} \equiv \langle u, v : u^2, v^2, (uv)^{2(2k+1)} \rangle = \langle u, uvv \rangle \times \langle (uv)^{2k+1} \rangle \equiv D_{2k+1} \times \mathbb{Z}_2$. If n is not of the form $2(2k+1)$ then D_n is irreducible.

Lemma 6 Suppose the group G decomposes as direct products $\prod_{i=1}^q A_i = \prod_{i=1}^q B_i$ where each A_i and B_i is either \mathbb{Z}_2 or D_k for $k \neq 2(2m+1)$, (i.e. D_k is an irreducible dihedral group). If $B_i = \langle x, y : x^2, y^2, (xy)^n \rangle$ then there exists a unique integer j such that $A_j = \langle u, v : u^2, v^2, (uv)^n \rangle$ and

(i) For odd n , $xy = (uv)^p$ and $\langle xy \rangle = \langle uv \rangle$

(ii) For even n , $xy = (uv)^p t$ where t is order 2 and commutes with u and v and $\langle (xy)^2 \rangle = \langle (uv)^2 \rangle$.

Proof: In either case, for all $t \in G$, $txyt^{-1} = (xy)^{\pm 1}$. Say $xy = a_1 \cdots a_q$ where $a_i \in A_i$. Since xy does not have order 2, we may assume that a_1 has order greater than 2. Say $A_1 = \langle u, v : u^2, v^2, (uv)^m \rangle$. Then $a_1 = (uv)^p$ and $(xy)^{\pm 1} = u(xy)u = a_1^{-1}a_2 \cdots a_q$. As $a_1 \neq a_1^{-1}$, we must have $a_i = a_i^{-1}$ for all $i \geq 2$. If xy has odd order, then each a_i is trivial or has odd order. In this case, $a_i = 1$ for all $i \geq 2$ and $xy = a_1$. If n is even then $xy = a_1t$ where t has order 2 and commutes with u and v .

In any case, the cyclic group $\langle xy \rangle$ is normal in G and the quotient of G by $\langle xy \rangle$ has irreducible decomposition obtained from $\prod_{j=1}^q B_j$ by replacing B_i by \mathbb{Z}_2 . Suppose n is odd, and $a_1 = (uv)^p$. By the Krull-Schmidt theorem (see [7]), $\langle (uv)^p \rangle$ must have index 2 in A_1 and so $\langle (uv)^p \rangle = \langle uv \rangle$ as desired. This implies that uv and $xy = (uv)^p$ have the same order and so $m = n$.

Now suppose n is even. In this case, $(xy)^2 = a_1^2 = (uv)^{2p}$. The quotient of G by the normal subgroup $\langle (xy)^2 \rangle$ has decomposition obtained from $\prod_{j=1}^q B_j$ by replacing B_i by $\mathbb{Z}_2 \times \mathbb{Z}_2$. Hence the quotient of A_1 by $\langle (uv)^{2p} \rangle$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$. If k is odd, then D_k does not map onto $\mathbb{Z}_2 \times \mathbb{Z}_2$ so uv has even order. If $(uv)^2 \notin \langle (uv)^{2p} \rangle$, then $1, u, uv, (uv)^2$ and $(uv)^3$ would represent 5 different $(uv)^{2p}$ -cosets of $\langle u, v \rangle$ which is impossible. Hence $\langle (uv)^{2p} \rangle = \langle (uv)^2 \rangle$. This implies that xy and uv have the same order and so $m = n$.

In either case $\langle (xy)^2 \rangle = \langle (uv)^2 \rangle$. The uniqueness follows by construction.

•

If a triangle $[xyz]$ in a diagram V for a Coxeter group has edge labels (a, b, c) , then $\langle x, y, z \rangle$ is finite if and only if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$. A result of Tits, implies that if A is a finite subgroup of a Coxeter group W , and (W, S) is a Coxeter system, then there is a subset T of S such that $\langle T \rangle$ is finite and A is a subgroup of a conjugate of $\langle T \rangle$. If (W, S) is even, and $T \subset S$ is such that $\langle T \rangle$ is finite, then $\langle T \rangle$ decomposes as a direct product of groups each factor of which is dihedral or \mathbb{Z}_2 . It is straightforward to see that the finite triangle groups $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$ are not isomorphic to a subgroup of a dihedral group D_n . The Krull-Schmidt theorem then implies that these finite triangle groups are not subgroups of an even Coxeter group.

If V is a diagram for a Coxeter system (W, S) and $[ab]$ is an even labeled edge of V , then a simple examination of presentations shows that $W/N((ab^2))$ is a Coxeter group with diagram obtained from V by changing the label of $[ab]$ to 2. The subgroup $\langle S - \{t\} \rangle$ for $t \in \{a, b\}$ injects under the quotient.

Proposition 7 *Suppose W is a finitely generated even Coxeter group with diagrams V and V' (not necessarily even).*

There is a unique bijection α between the edges $[xy]$ of V' with label > 2 and the edges $[ab]$ of V with label > 2 such that if $\alpha([xy]) = [ab]$, then exactly one of the following holds.

- (i) $[xy]$ and $[ab]$ are labeled $2k + 1$, xy is conjugate to $(ab)^p$ for some integer p and the cyclic group $\langle xy \rangle$ is conjugate to $\langle ab \rangle$
- (ii) $[xy]$ is labeled $2k + 1$ and $[ab]$ is labeled $2(2k + 1)$, xy is conjugate to $(ab)^{2p}$ and the cyclic group $\langle xy \rangle$ is conjugate to $\langle (ab)^2 \rangle$.
- (ii') $[xy]$ is labeled $2(2k + 1)$ and $[ab]$ is labeled $2k + 1$, ab is conjugate to $(xy)^{2p}$ and the cyclic group $\langle (xy)^2 \rangle$ is conjugate to $\langle ab \rangle$.
- (iii) $[xy]$ and $[ab]$ are labeled $2(2k + 1)$, $(xy)^2$ is conjugate to $(ab)^{2p}$ and the cyclic group $\langle (xy)^2 \rangle$ is conjugate to $\langle (ab)^2 \rangle$.
- (iv) $[xy]$ and $[ab]$ are labeled $4n$, $(xy)^2$ is conjugate to $(ab)^{2p}$ and the cyclic group $\langle (xy)^2 \rangle$ is conjugate to $\langle (ab)^2 \rangle$.

Proof: Observe that the Proposition can be reduced to showing parts (ii), (iii) and (iv) are valid when all edges of V have even labels. So we make that assumption. Let σ' be a maximal spherical simplex of V' containing $[xy]$ and σ the maximal spherical simplex of V such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$. By conjugating, we assume $\langle \sigma' \rangle = \langle \sigma \rangle$. Let e'_1, \dots, e'_n be the edges of $\langle \sigma' \rangle$ not labeled by 2. Then $\langle \sigma' \rangle$ naturally decomposes as a direct product $A' \cong \prod_{i=1}^m A'_i$ and for $i \in \{1, \dots, n\}$, A'_i is the dihedral group D_{k_i} where k_i is the label of e'_i when e'_i has odd label or label a multiple of 4, and k_i is half the label of e'_i if e'_i has label two times an odd. So if $e'_i = [xy]$ has label 2 times an odd integer q , then $q = k_i$ and $D_{2q} = \langle x, y \rangle$ decomposes as in Remark 1 as $\mathbb{Z}_2 \times D_q = \langle (xy)^q \rangle \times \langle x, yxy \rangle$. All other A'_j are copies of \mathbb{Z}_2 . Similarly decompose $\langle \sigma \rangle$ as the direct product $A \cong \prod_{i=1}^m A_i$. (Note that the number of factors in the decompositions of $\langle \sigma \rangle$ and $\langle \sigma' \rangle$ are the same and there is a bijection ϕ of the set of A'_i to the set of A_i such that $\phi(A'_i)$ is isomorphic to A_i , by Kurl-Schmidt.)

Now apply Lemma 6 to get a map α from the edges with label > 2 of V' to those of V . If $[ab]$ and $[cd]$ are distinct edges of V with labels > 2 , then $\langle (ab)^2 \rangle$ is not conjugate to $\langle (cd)^2 \rangle$ since $\langle (cd)^2 \rangle$ injects under the quotient of W by $N((ab)^2)$ (the normal closure of $(ab)^2$ in W). Hence there is exactly one choice for α . By considering maximal spherical simplices in V , we see that α is onto.

It remains to show that α is injective. First we prove the several lemmas.

Lemma 8 *Suppose V and V' are diagrams for an even Coxeter group W and V is even. Then no triangle of V' , containing an edge with odd label, contains two edges corresponding to distinct edges of V . Equivalently, if $[xyz]$ is a triangle of V' and some edge of $[xyz]$ has odd label and corresponds to the edge $[ab]$ of V then either all edges of $[xyz]$ have odd label and correspond to $[ab]$, or two edges of $[xyz]$ are labeled 2, or two edges of $[xyz]$ have odd labels and correspond to $[ab]$ and the other edge is labeled 2.*

Proof: Suppose $[xy]$ is labeled $2k + 1$ and corresponds to the edge $[ab]$ of V . Also assume that $[xz]$ is labeled $m > 2$ and corresponds to $[cd] \neq [ab]$. Then either $[zy]$ is labeled 2, or $[zy]$ corresponds to $[ab]$, or $[zy]$ corresponds to $[cd]$, or $[zy]$ corresponds to $[ef] \notin \{[cd], [ab]\}$. Observe that $N(xy) = N((ab)^2)$. We consider the quotient map $q : W \rightarrow W/N(xy)$. Now, $\langle c, d \rangle$ injects under q , but if $[zy]$ is labeled 2, then $(xz)^2 \in \ker(q)$. Either $\langle xz \rangle$ is conjugate to $\langle (cd)^2 \rangle$ and $(cd)^2$ has odd order, or $\langle (xz)^2 \rangle$ is conjugate to $\langle (cd)^2 \rangle$. In the first case, $(cd)^4$ is in $\ker(q)$ which is impossible. In the second case, $(cd)^2$ is in $\ker(q)$ which is also impossible.

If $[zy]$ corresponds to $[ab]$ then $[zy]$ has label $2k + 1$ or $2(2k + 1)$. This label cannot be $2(2k + 1)$ since otherwise, $N((zy)^2) = N((ab)^2) = N(xy)$, but $xy \notin N((zy)^2)$. This label cannot be $2k + 1$, since otherwise, $N(zy) = N((ab)^2) = N(xy)$. But, $xy \in N(yz)$ implies $xz \in N(yz)$. This implies $(cd)^2 \in N(yz) = N((ab)^2)$. But $(cd)^2 \notin N((ab)^2)$.

If $[yz]$ corresponds to $[cd]$, then assume that $[cd]$ has label $2n$. The labels of $[xz]$ and $[yz]$ are $2n$ or n . If $[xz]$ is labeled $2n$, then $N((xz)^2) = N((cd)^2)$ is equal to $N((yz)^2)$ or $N(yz)$, but $(yz), (yz)^2 \notin N((xz)^2)$. Hence $[xz]$ and similarly $[yz]$ is labeled n . Then $N(xz) = N((cd)^2) = N(zy)$. If $zy \in N(xz)$, then $xy \in N(xz)$ implying $N(xy) \equiv N((ab)^2) < N(xz) = N((cd)^2)$. But, $(ab)^2 \notin N((cd)^2)$.

If $[yz]$ corresponds to $[ef]$, then let $q : W \rightarrow W/N(xy) = W/N((ab)^2)$ be the quotient map. We have $q(zx) = q(zy)$. If the labels of $[xz]$ and $[zy]$ are odd, then $q(\langle (cd)^2 \rangle) = q(\langle xz \rangle) = q(\langle yz \rangle) = q(\langle (ef)^2 \rangle)$. But this is impossible as $q(\langle (cd)^2 \rangle) \neq q(\langle (ef)^2 \rangle)$ in $W/N((ab)^2)$. If the labels of $[xz]$ and $[zy]$ are even, then $q(\langle (cd)^2 \rangle) = q(\langle (xz)^2 \rangle) = q(\langle (zy)^2 \rangle) = q(\langle (ef)^2 \rangle)$ which is again impossible. If the label of $[xz]$ is odd and the label of $[zy]$ is even then $q(\langle (cd)^4 \rangle) = q(\langle (xz)^2 \rangle) = q(\langle (zy)^2 \rangle) = q(\langle (ef)^2 \rangle)$. Again, $q(\langle (cd)^4 \rangle) \neq q(\langle (ef)^2 \rangle)$ in $W/N((ab)^2)$. Similarly if the label of $[xz]$ is even and the label of $[zy]$ is odd.

We conclude that each edge of $[xyz]$ with label > 2 corresponds to $[ab]$.

If $[xz]$ has even label > 2 , then $N((xz)^2) = N((ab)^2) = N(xy)$ but $xy \notin N((xz)^2)$. •

Now we improve Lemma 8

Lemma 9 *Suppose W is a finitely generated even Coxeter group. If V' is a diagram for W with odd labeled edge $[xy]$ then any triangle of V' containing $[xy]$ has two edges labeled 2.*

Proof: Suppose V' is a diagram for a minimal (with respect to $T(V') \equiv$ the product of all edge labels of V') counterexample to the Lemma and V is an even diagram for W . By Lemma 8, we may assume $[xyz]$ is a triangle of V' with at least 2 odd labeled edges corresponding to $[ab]$ in V . Let σ' be a maximal simplex containing $[xyz]$. Then by [6], $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for σ a maximal simplex of V . By minimality, $V' = \sigma'$, so we have V' and V are complete. Let δ' be the set of all vertices in V' that belong to an edge that corresponds to $[ab]$. If $[st]$ is an edge of V' such that $s \in \delta'$ and $t \notin \delta'$, then by Lemma 8, $[st]$ has label 2. I.e. V' decomposes as $\langle \delta' \rangle \times \langle V' - \delta' \rangle$. If any edge $[st]$ of V' were labeled by an even > 2 , then $W/N((st)^2)$ would be a smaller counterexample. If $s, t \in V' - \delta'$ and $[st]$ is an edge with odd label, then $N(st) \subset N(V' - \delta')$ and so $\langle \delta' \rangle$ injects under the quotient map $W \rightarrow W/N(st)$. But then $W/N(st)$ is a smaller counterexample. Hence $[ab]$ is the only edge of V not labeled 2. I.e. W is finite. This is impossible as a $(2, n, n)$ triangle group is infinite unless $n = 2$ or 3 . We previously ruled out $(2, 3, 3)$ triangle groups as subgroups of even Coxeter groups. •

As a direct consequence of the previous lemma we have the following.

Lemma 10 *Suppose W is a finitely generated even Coxeter group and V' is a diagram with odd labeled edge $[xy]$. Then the diagram for $W/N(xy)$ obtained from V' by collapsing the edge $[xy]$ is such that no other edge of V' is collapsed and the only edges of V' that are identified are those in a triangle containing $[xy]$. •*

It is now elementary to show the injectivity of α . If distinct edges $[xy]$ and $[zw]$ of V' correspond to $[ab]$ in V , then a quotient argument easily implies neither $[xy]$ nor $[zw]$ has even label. But then $N(xy) = N((ab)^2) = N(zw)$. But by Lemma 10, $zw \notin N(xy)$. •

Example 1. The element cba of the group $\langle a, b, c : a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ac)^2 = 1 \rangle$ conjugates b to c and a to b . Indicating the need for a more sophisticated version of Proposition 7 in a more general setting.

The Deletion Condition for Coxeter groups implies the following Lemma.

Lemma 11 *Suppose (W, S) is a Coxeter system, Γ the Cayley graph of W with respect to S and $T \subset S$. If u and v are vertices of Γ (i.e. elements of W), then there is a unique closest vertex w of the coset $v\langle T \rangle$ to u . Furthermore, if α is a geodesic from u to w , and β is a geodesic at w in the letters of T , then $\alpha\beta$ is geodesic. •*

Proposition 12 *Suppose (W, S) is an even Coxeter system, $a, b \in S$ and ab has finite order > 2 . If $y \in W$ is such that y conjugates $(ab)^2$ to $(ab)^{\pm 2}$ then y can be written geodesically as uv where $u \in \langle a, b \rangle$ and $v \in lk_2(a, b)$.*

Proof: We first show that y conjugates $\langle a, b \rangle$ to itself. We have $(ab)^2 \in \langle a, b \rangle \cap y\langle a, b \rangle y^{-1} = v\langle T \rangle v^{-1}$ for $T \subset \{a, b\}$ and $v \in \langle a, b \rangle$. If T is a single element, then $(ab)^2$ is conjugate to a or b . This is impossible as $(ab)^2$ has even length. Hence $T = \langle a, b \rangle$ and so $\langle a, b \rangle = y\langle a, b \rangle y^{-1}$.

Let Γ be the Cayley graph of W with respect to S . Write $y = x_1 y_1 x_2$ where $x_i \in \langle a, b \rangle$ and y_1 is the shortest element of the double coset $\langle a, b \rangle y \langle a, b \rangle$. We show that y_1 commutes a and b . If α is a geodesic in Γ from 1 to y_1 then $a\alpha, b\alpha, \alpha a$ and αb are geodesic by the choice of y_1 . Hence by Lemma 11 if β_1 and β_2 are geodesic paths at 1 and y_1 respectively, in the letters a, b , then the paths (β_1^{-1}, α) and (α, β_2) are geodesic. Now, since $y_1 a y_1^{-1}$ and $y_1 b y_1^{-1}$ are in $\langle a, b \rangle$ they must both be of length 1. I.e. (since (W, S) is even) y_1 commutes with a and b . Furthermore, $y = x_1 x_2 y_1$. Results in [3] and [2] implies y_1 is a product of an element of $\langle a, b \rangle$ and an element of $lk_2(a, b)$. •

Remark 2 Each part of Proposition 7 concludes that xy or $(xy)^2$ is conjugate to $(ab)^p$ or $(ab)^{2p}$. Hence if c is the conjugating element, then $cab c^{-1}$ commutes with $(xy)^2$ and by Proposition 12, $cab c^{-1} = uv$ for $u \in \langle x, y \rangle$ and $v \in lk_2(x, y)$. Similarly, $c^{-1} x y c$ is conjugate to $u'v'$ for $u' \in \langle a, b \rangle$ and $v' \in lk_2(a, b)$. Hence, parts (ii'), (iii) and (iv) of Proposition 7 can be improved to say: (xy) is conjugate to $(ab)^p t$, where t commutes with a and b and $t^2 = 1$. Parts (ii), (iii) and (iv) can be improved to say: (ab) is conjugate to $(xy)^q s$, where s commutes with x and y and $s^2 = 1$.

Proposition 13 *Suppose (W, S) is an even Coxeter system with diagram V , (W, S') is another Coxeter system with diagram V' , $[xy]$ and $[yz]$ are distinct edges of V' with labels > 2 . If $[ab]$ and $[cd]$ are edges of V which correspond to $[xy]$ and $[yz]$ respectively, then $\{a, b\} \cap \{c, d\}$ contains exactly one element. I.e. The edges $[ab]$ and $[cd]$ share exactly one vertex.*

Proof: Suppose that $[xy]$ has label $k > 2$ and $[yz]$ has label $m > 2$. Observe that $[ab]$ is labeled l where $l = k$ or $l = 2k$ and $[cd]$ by n where $n = m$ or $n = 2m$. The set $\{a, b\} \cap \{c, d\} \neq \{a, b\}$ by the uniqueness of pairing of Proposition 7. Assume that $\{a, b\} \cap \{c, d\} = \emptyset$. Now xy or $(xy)^2$ is equal to $w(ab)^{2p}w^{-1}$ for some $w \in W$. By conjugation we may assume that either yz or $(yz)^2$ is equal to $(cd)^{2r}$.

Lemma 14 *Not both a and b commute with both c and d .*

Proof: Otherwise there is a maximal spherical simplex σ (i.e. $\langle \sigma \rangle$ is finite) containing a, b, c and d . The group $\langle \sigma \rangle$ is conjugate to $\langle \sigma' \rangle$ for σ' a simplex of V' containing x, y and z , which is impossible as $\langle x, y, z \rangle$ is not finite. •

Consider the retraction $\alpha : W \rightarrow \langle a, b, c, d \rangle \equiv G$ with kernel $N(S - \{a, b, c, d\})$.

Lemma 15 *Either a or b is an element of $lk_2(c, d)$ and either c or d is an element of $lk_2(a, b)$.*

Proof: Let $\alpha(x) \equiv \bar{x}$, $\alpha(y) \equiv \bar{y}$, $\alpha(z) \equiv \bar{z}$ and $\alpha(w) \equiv \bar{w}$. As $(ab)^{2p} = w^{-1}(xy)w$ or $w^{-1}(xy)^2w$, Proposition 12 implies $\bar{w}^{-1}\bar{y}\bar{w} = u_1v_1$, where $u_1 \in \langle a, b \rangle$ and $v_1 \in lk_2(a, b)$ (where lk_2 is taken in $\langle a, b, c, d \rangle$), and $\bar{y} = u_2v_2$ where $u_2 \in \langle c, d \rangle$ and $v_2 \in lk_2(c, d)$. Note that $v_1 = c$ or d or 1 and $v_2 = a$ or b or 1 . If $v_2 = 1$, then $\bar{y} \in \langle c, d \rangle$. The element $\bar{w}^{-1}\bar{y}\bar{w}$ conjugates $(ab)^2$ to $(ba)^2$. As $\bar{y} (\in \langle c, d \rangle)$ is in the kernel of the retraction of $\langle a, b, c, d \rangle$ to $\langle a, b \rangle$ (with kernel $N(\{c, d\})$), this is impossible unless $(ab)^2 = (ba)^2$. If ab has order 4, then xy has order 4 and Remark 2 implies $xyt = w(ab)^qw^{-1}$ where t has order 2 and commutes with x and y . In this case we see that $w^{-1}\bar{y}w$ conjugates ab to ba . Again this is impossible as $\bar{y} \in \langle c, d \rangle$ is in the kernel of a retraction of $\langle a, b, c, d \rangle$ to $\langle a, b \rangle$ and $ab \neq ba$. Similarly, $v_1 \neq 1$. •

Without loss, we assume that $a \in lk_2(c, d)$ and $c \in lk_2(a, b)$. (See Figure 1.)

Recall, $T(V')$ is the product of the edge labels of V' . Assume that V' is a minimal (under T) counterexample to Proposition 13.

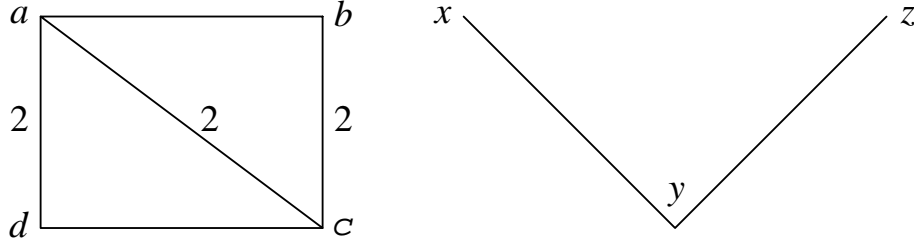


Figure 1:

Lemma 16 *There is no edge $[bd]$.*

Proof: Otherwise, $[bd]$ has label > 2 by Lemma 14. If the edge $[st]$ of V' corresponds to $[bd]$ and st has even label $2k$, then $[bd]$ has label $2k$ and $W/N((bd)^2)$ is a smaller counterexample to our proposition. Hence we may assume that $[st]$ has odd label. By Lemma 9, $\{s, t\} \neq \{x, z\}$, and by Lemma 10, $W/N(st) = W/N((bd)^2)$ is a smaller counterexample. •

Lemma 17 *There is no edge $[xz]$.*

Proof: Otherwise, let σ' be a maximal simplex of V' containing the triangle $[xyz]$. By [6], there is a simplex σ of V such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$. Thus, $\{a, b, c, d\} \subset \sigma$ contradicting Lemma 16. •

Lemma 18 *In V' , the only edges not labeled 2 are $[xy]$, $[yz]$.*

Proof: By the minimality assumption, there are no edges of V' with even label > 2 . If $[st]$ is distinct from $[xy]$ and $[yz]$, and with odd label, then the quotient of W by $N(st)$ gives a smaller counterexample by Lemma 10. •

Let S' be the vertex set of V' . Let λ be the retraction of W to $\langle x, y, z \rangle$ with kernel $N(S' - \{x, y, z\})$. Observe that the groups $\langle b, c \rangle$, $\langle a, d \rangle$, and $\langle a, c \rangle$ inject under λ , for otherwise $(ab)^2$ or $(cd)^2$ is in $\ker(\lambda)$ (see Figure 1) which they are not.

Now $\langle x, y, z \rangle = \langle x, y \rangle *_{\langle y \rangle} \langle y, z \rangle$. A maximal simplex σ of V containing $\{a, b, c\}$ is such that $\langle \sigma \rangle$ is conjugate to a $\langle \sigma' \rangle$ for σ' a maximal simplex of V' . Hence $\lambda(\langle a, b, c \rangle)$ is a subgroup of a conjugate of $\langle x, y \rangle$ or $\langle y, z \rangle$. As $\lambda((ab)^2)$ is in the kernel of the quotient of $\langle x, y, z \rangle$ by $N(xy)$, and $\langle y, z \rangle$ injects under

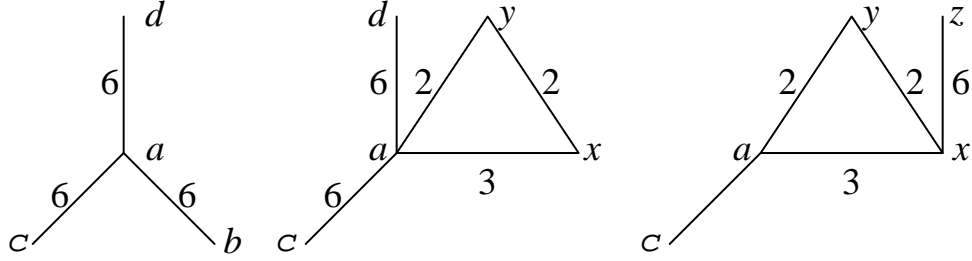


Figure 2:

this quotient, $\lambda(\langle a, b, c \rangle)$ is a subgroup of a conjugate of $\langle x, y \rangle$. Similarly, $\lambda(\langle a, c, d \rangle)$ is a subgroup of a conjugate of $\langle y, z \rangle$. Hence (simply consider edge and vertex stabilizers of the Bass-Serre tree for $\langle x, y \rangle *_{\langle y \rangle} \langle y, z \rangle$) $\lambda(\langle a, c \rangle)$ is a subgroup of a conjugate of $\langle y \rangle$. But this is impossible as $\langle y \rangle$ has order 2. •

Example 2 Three diagrams for an even Coxeter group are shown in Figure 2. Isomorphisms between presentations determined by these diagrams are given by:

$a \rightarrow a, c \rightarrow c, d \rightarrow d, x \rightarrow bab$ and $y \rightarrow (ab)^3$ (a triangle/edge exchange) and $a \rightarrow a, c \rightarrow c, x \rightarrow x, y \rightarrow y$ and $z \rightarrow axadaxa$ (a twist of $\{d\}$ around $[ax]$). The correspondence of Proposition 7 between edges of the first and last diagram of Figure 2, match the adjacent edges $[ac]$ and $[ad]$ with the non-adjacent edges $[ac]$ and $[xz]$, respectively.

Proposition 19 *Suppose (W, S) is an even Coxeter system with diagram V , (W, S') is another Coxeter system with diagram V' , $[ab]$ and $[ac]$ are distinct edges of V with labels > 2 . If $[xy]$ and $[zv]$ are edges of V' which correspond to $[ab]$ and $[ac]$ respectively, then either there is an odd edge path in V' connecting a vertex of $\{x, y\}$ and a vertex of $\{z, v\}$, or $\{x, y\} \cap \{z, v\}$ contains exactly one element.*

Proof: Suppose there is no odd edge path in V' with one vertex in $\{x, y\}$ and one vertex in $\{z, v\}$, $\{x, y\} \cap \{z, v\} = \emptyset$, and V' is a minimal (over $T(V')$) such counterexample to our proposition.

Lemma 20 *The only edges of V' not labeled 2 are $[xy]$ and $[zv]$.*

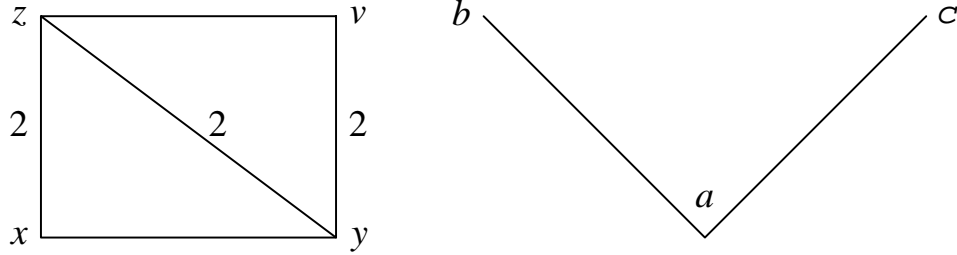


Figure 3:

Proof: By the minimality of $T(V')$, there is no edge of V' with a label that is even and > 2 , other than possibly $[xy]$ and $[zv]$. Also by the minimality of $T(V')$ and Lemma 10 there is no edge of V' with odd label other than $[xy]$ or $[zv]$. •

By Lemma 20 and Proposition 7, the only edges of V not labeled 2 are $[ab]$ and $[ac]$. Let σ be a maximal spherical simplex of V containing $\{a, b\}$ and σ' be a maximal spherical simplex in V' such that $\bar{w}\langle\sigma'\rangle\bar{w}^{-1} = \langle\sigma\rangle$ for some $\bar{w} \in W$. Then $\{x, y\} \subset \sigma'$. If neither z nor w is an element of σ' then $W/N(\sigma) = W/N(\sigma')$ is a Coxeter group with diagram obtained from V (respectively V') by removing the vertices of σ (respectively σ'). But one of these diagrams has all edges labeled 2 and the other has an edge with label > 2 , which is impossible. We conclude that v or z is an element of σ' and so v or z is an element of $lk_2(x, y)$. Similarly, x or y is an element of $lk_2(z, v)$. Say $z \in lk_2(x, y)$ and $y \in lk_2(z, v)$. See Figure 3.

Consider the retraction τ of W to $\langle a, b, c \rangle$ with kernel $N(S - \{a, b, c\})$. By Proposition 7, $\{(xy)^2, (zv)^2\} \cap \ker(\tau) = \emptyset$. Hence $\{x, y, z, v, zy, xz, yv\} \cap \ker(\tau) = \emptyset$ (see Figure 3). There is no edge $[bc]$ as every simplex of V' (and hence every simplex of V) is spherical. Observe that $\langle a, b, c \rangle = \langle a, b \rangle *_{\langle a \rangle} \langle a, c \rangle$. Note that $\tau(xy)$ has order > 2 , and is an element of a conjugate of $\langle a, b \rangle$. Hence $\tau(xy)$ cannot be an element of distinct conjugates of $\langle a, b \rangle$, or some conjugate of $\langle a, c \rangle$, since otherwise, (simply consider vertex and edge stabilizers of the Bass-Serre tree for $\langle a, b \rangle *_{\langle a \rangle} \langle a, c \rangle$) $\tau(xy)$ is an element of a conjugate of $\langle a \rangle$, an order 2 group. Therefore, $\tau(\langle x, y, z \rangle)$ is a subgroup of a conjugate of $\langle a, b \rangle$ and $\tau(\langle y, z, v \rangle)$ is a subgroup of a conjugate of $\langle a, c \rangle$. But this implies (again consider vertex and edge stabilizers of the Bass-Serre tree for $\langle a, b \rangle *_{\langle a \rangle} \langle a, c \rangle$) that $\alpha(\langle z, y \rangle)$ is a subgroup of a conjugate of $\langle a \rangle$. This is impossible and the proof of the proposition is complete. •

The following result is used in [1], so we cannot use [1] to simplify the proof.

Lemma 21 *Suppose V is an even diagram for the finitely generated Coxeter group W and V' is another diagram for W . If $[abc]$ is a triangle of V having at least two edges with label > 2 , then the edges of $[abc]$ with label > 2 correspond to edges with label > 2 of a triangle $[xyz]$ of V' .*

Proof: Let σ be a maximal simplex of V containing $[abc]$. By [6], there is a maximal simplex σ' in V' such that $\langle \sigma \rangle$ is conjugate to $\langle \sigma' \rangle$. By Proposition 7 applied to σ and σ' and the uniqueness conclusion of Proposition 7 applied to V and V' , σ' contains the edges of V' corresponding to those of $[abc]$ that have label > 2 . First we show the edges of σ' with label > 2 and corresponding to those of $[abc]$ with label > 2 are mutually adjacent. If not, say $[xy]$ and $[zv]$ are two such non-adjacent edges. By Proposition 19 (applied to the even Coxeter group $\langle \sigma \rangle$) there is an odd labeled edge (in σ') adjacent to $[xy]$, but this is impossible by Lemma 9.

If an edge of $[abc]$ is labeled 2, we are finished. Otherwise, the edges of $\langle \sigma' \rangle$ corresponding to those of $[abc]$ must have even labels by Lemma 9 and either form a triangle or triad. If a triad is formed, then we may assume that $[ab]$ corresponds to $[xy]$ in V' , $[bc]$ corresponds to $[xz]$ and $[ac]$ corresponds to $[xv]$. Now assume that V is a minimal (with respect to $T(V)$) counterexample to the Lemma. Then $V = \sigma$ and $V' = \sigma'$. By minimality, every edge of V except $[ab]$, $[bc]$ and $[ac]$ has label 2. Similarly for V' . In particular, V and V' are even. By conjugation, we may assume that $\langle (xy)^2 \rangle = \langle (ab)^2 \rangle$. As x conjugates $(ab)^2$ to $(ba)^2$, Proposition 12 implies that, $x = u_1 t_1$ where $u_1 \in \langle a, b \rangle$ and $t_1 \in lk_2(a, b)$. Similarly, $x = w_2 u_2 t_2 w_2^{-1}$ where $w_2 \in W$, $u_2 \in \langle b, c \rangle$ and $t_2 \in lk_2(b, c)$ and $x = w_3 u_3 t_3 w_3^{-1}$ where $w_3 \in W$, $u_3 \in \langle a, c \rangle$ and $t_3 \in lk_2(a, c)$.

Now $x \in (\langle \{a, b\} \cup lk_2(a, b) \rangle) \cap (w_2 \langle \{b, c\} \cup lk_2(b, c) \rangle w_2^{-1}) \cap (w_3 \langle \{a, c\} \cup lk_2(a, c) \rangle w_3^{-1}) = v \langle T \rangle v^{-1}$ where $T \subset \{a, b\} \cup lk_2(a, b)$. Clearly, $c \notin T$. As no conjugate of a is an element of $w_2 \langle \{b, c\} \cup lk_2(b, c) \rangle w_2^{-1}$, $a \notin T$. Similarly, $b \notin T$. But then T is central in W , implying x is central, the desired contradiction. •

Lemma 22 *Suppose V is an even diagram for the finitely generated Coxeter group W and V' is another diagram for W . If $[xyz]$ is a triangle of V' having at least two edges with label > 2 , then the edges of $[xyz]$ with label > 2 correspond to edges with label > 2 of a triangle $[abc]$ of V .*

Proof: By Lemma 9 each edge of $[xyz]$ has even label. Suppose V' is a minimal (with respect to $T(V')$) counterexample. By Lemma 10, V' contains no odd labeled edge, and so V' is even. Now apply Lemma 21. •

Remark 3 Observe in each of the last two lemmas that if we begin with a triangle with exactly one edge labeled 2, we do not conclude that the corresponding triangle has an edge labeled 2. This can now be resolved by combining the last two lemmas. More specifically, we see that if W is an even Coxeter group, V and V' are diagrams for W and $[abc]$ is a triangle of V with only one edge with label 2, then there is a corresponding triangle $[xyz]$ of V' with only one edge labeled 2. If $[abc]$ has no edge labeled 2 then $[xyz]$ has no edge labeled 2.

As a direct application of Lemma 21 and Lemma 22 we have an analogue for Lemma 9.

Lemma 23 *Suppose W is an even Coxeter group, V and V' are diagrams for W , $[xy]$ is an odd labeled edge of V' and $[ab]$ the edge of V corresponding to $[xy]$. Then any triangle containing $[ab]$ has two edges labeled 2. •*

Proposition 24 *Suppose V is an even diagram for the finitely generated Coxeter group W and V' is another diagram for W . If $[xy]$ is an odd labeled edge of V' and $[ab]$ is the corresponding edge of V , then with the exception of $[ab]$, every edge adjacent to a is labeled 2 or every edge adjacent to b is labeled 2.*

Proof: Suppose $[xy]$ has label $2k + 1$. It suffices to show:

Lemma 25 *There is no edge path $([ca], [ab], [bd])$ in V such that each edge has label > 2 .*

Proof: Since $[xy]$ has odd label, Lemma 23 implies the path $([ca], [ab], [bd])$ does not form a triangle. Suppose that $[ca]$ and $[bd]$ correspond to $[uv]$ and $[st]$ respectively in V' . By Proposition 19, select a shortest path with odd labeled edges, e_1, \dots, e_n from $\{u, v\}$ to $\{x, y\}$. Assume that $e_i = [x_{i-1}x_i]$ for all i . If $e_j = [st]$, then a diagram \bar{V}' for $W/N(\{x_0x_1, \dots, x_{j-2}x_{j-1}\})$ is obtained from V' by collapsing each edge of the set $\{e_1, \dots, e_{j-1}\}$. The corresponding diagram \bar{V} for $W/N(\{x_0x_1, \dots, x_{j-2}x_{j-1}\})$, is obtained from V by replacing each label of an edge corresponding to one of $\{e_1, \dots, e_{j-1}\}$ by 2. In \bar{V}' , $[uv]$ and $[st]$ are adjacent, but $[ca]$ and $[bd]$ are not, contradicting Proposition

13. We conclude that $[st]$ is not in $\{e_1, \dots, e_n\}$. Similarly, if d_1, \dots, d_m is a shortest odd labeled edge path from $\{x, y\}$ to $\{s, t\}$, we may assume that $[uv]$ is not in $\{d_1, \dots, d_m\}$. Assume that $d_i = [y_{i-1}y_i]$ for all i . A diagram \tilde{V}' is obtained for the group $\tilde{W} \equiv W/N(\{x_0x_1, \dots, x_{n-1}x_n, xy, y_0y_1, \dots, y_{m-1}y_m\})$ by collapsing the edges $e_1, \dots, e_n, [xy], d_1, \dots, d_m$ of V' . Hence in \tilde{V}' , $[uv]$ and $[st]$ are adjacent. Let \tilde{V} be the diagram for \tilde{W} obtained from V by changing the edge labels of the edges of V corresponding to $e_1, \dots, e_n, [xy], d_1, \dots, d_m$ to 2. In \tilde{V} , $[ca]$ and $[bd]$ are not adjacent, contradicting Proposition 13. • •

The following theorem classifies even rigid Coxeter groups.

Theorem 26 *If V is an even diagram for the Coxeter group W then W has a diagram that is not even if and only if there is an edge $[ab]$ in V with label $2(2k + 1)$ for $k > 0$, such that with the exception of $[ab]$, every edge of V containing a is labeled 2 and if $[ac]$ is such an edge, then there is an edge $[bc]$ with label 2.*

Proof: If $[ab]$ is an edge as described in the theorem, then $\langle a, b : a^2 = b^2 = (ab)^{2(2k+1)} = 1 \rangle$ is isomorphic to the group $\langle x, y, z : x^2 = y^2 = z^2 = (xz)^2 = (yz)^2 = (zy)^{2k+1} = 1 \rangle$ by the map extending $x \rightarrow a$, $y \rightarrow bab$ and $z \rightarrow (ab)^{2k+1}$. It is elementary to see that the edge $[ab]$ in V can be replaced by the triangle $[xyz]$ to give a new diagram for W .

The proof of the converse is more delicate. Recall that $T(V)$ is the product of all edge labels in V . We assume from this point on that V is a minimal (with respect to T) counterexample to our theorem. Let V' be a diagram for W with odd labeled edge $[xy]$. Assume that $[xy]$ corresponds to the edge $[ab]$ of V .

Lemma 27 *With the exception of $[ab]$ every edge of V is labeled 2 (and hence $[xy]$ is the only edge of V' not labeled 2.)*

Proof: If $[cd]$ is an edge labeled $n > 2$ and $\{c, d\} \cap \{a, b\} = \emptyset$, then the quotient of W by $N((cd)^2)$ is a “smaller” counterexample. If $[ac]$ has label $n > 2$, then there is no edge $[bc]$ by Lemma 23. Again the quotient of W by $N((ac)^2)$ is a smaller counterexample. Similarly, there is no edge $[bc]$ with label > 2 . •

Lemma 28 *Suppose σ and σ' are simplices of V and V' respectively such that $\langle \sigma \rangle$ is conjugate to $\langle \sigma' \rangle$. Then σ contains $\{a, b\}$ if and only if σ' contains $\{x, y\}$. Also, σ contains exactly one element of $\{a, b\}$ if and only if σ' contains exactly one element of $\{x, y\}$.*

Proof: The group $\langle \sigma \rangle$ (respectively $\langle \sigma' \rangle$) is non-abelian if and only if $\{a, b\} \subset \sigma$ (respectively $\{x, y\} \subset \sigma'$). Hence the first conclusion of the lemma follows.

The group $W/N(\sigma)$ (respectively $W/N(\sigma')$) is abelian iff a or $b \in \sigma$ (respectively x or $y \in \sigma'$). But $W/N(\sigma) = W/N(\sigma')$. •

To finish Theorem 26, it suffices to show that V cannot contain edges $[ad]$ and $[bc]$ ($d \neq b$ and $c \neq a$) such that there is no edge between b and d and no edge between a and c . (There may or may not be an edge $[cd]$.) Assume otherwise. Let $\sigma(a, d)$ and $\sigma(a, b)$ be maximal simplices of V containing $\{a, d\}$ and $\{a, b\}$ respectively. Let $\sigma = \sigma(a, d) \cap \sigma(a, b)$. Note that $a \in \sigma$, but $\{b, c, d\} \cap \sigma = \emptyset$. By conjugation we may assume that $\sigma'(a, b)$ is a maximal simplex of V' such that $\langle \sigma'(a, b) \rangle = \langle \sigma(a, b) \rangle$ and that $w\langle \sigma'(a, d) \rangle w^{-1} = \langle \sigma(a, d) \rangle$ for $\sigma'(a, d)$ a maximal simplex of V' and $w \in W$. Then $\langle \sigma \rangle = \langle \sigma'(a, b) \rangle \cap w\langle \sigma'(a, d) \rangle w^{-1} = v\langle T \rangle v^{-1}$, for some $v \in \langle \sigma'(a, b) \rangle$, and $T \subset \sigma'(a, b)$. By Lemma 28 either x or y , but not both is an element of T .

Let q be the retraction of W to $\langle a, b, c, d \rangle$ with kernel $N(S - \{a, b, c, d\})$. Then, $q(\langle a \rangle) = q(\langle \sigma(a, b) \rangle \cap \langle \sigma(a, d) \rangle) = q(v\langle T \rangle v^{-1})$. Observe that x is conjugate to y , $\langle xy \rangle$ is conjugate to $\langle (ab)^2 \rangle$ and $q(ab)$ has order $2(2k+1)$. Thus, $q(x) \neq 1 \neq q(y)$ and so $q(\langle a \rangle)$ is conjugate to $q(\langle x \rangle)$ and $q(\langle y \rangle)$. Hence $q(a)$ is conjugate to $q(x)$ and $q(y)$. Similarly for b . This implies $q(a)$ and $q(b)$ are conjugate, the desired contradiction. Theorem 26 is finished. •

5 The Proof of Proposition 5

Lemma 29 *Suppose $\langle \sigma' \rangle = w\langle \sigma \rangle w$ for σ' a simplex of V' , σ a simplex of V and $w \in W$. If $f : \langle \sigma' \rangle \rightarrow \mathbb{Z}_2 \equiv \{-1, 1\}$ is a homomorphism, let N be the normal closure in W of $\ker(f)$. Then W/N is an even Coxeter group with diagram obtained from V by removing the vertices of $\sigma_1 \equiv \{s \in \sigma : f(wsw^{-1}) = 1\}$ and identifying the vertices of $\sigma - \sigma_1$.*

Proof: The kernel of f is generated by K' , the normal closure in $\langle \sigma' \rangle$ of $\{s \in \sigma' : f(s) = 1\} \cup \{st : s, t \in \sigma', f(s) = f(t) \neq 1\}$. Hence N is the normal closure of K' in W .

As $\langle \sigma' \rangle = w\langle \sigma \rangle w^{-1}$, K' can also be described as the normal closure in $w\langle \sigma \rangle w^{-1}$ of $K = \{wsw^{-1} : s \in \sigma \text{ and } f(wsw^{-1}) = 1\} \cup \{wstw^{-1} : s, t \in \sigma \text{ and } f(wsw^{-1}) = f(wtw^{-1}) \neq 1\}$. Now, in W , the normal closure of K' , K and $w^{-1}Kw$ are the same. •

Lemma 30 *Suppose (W, S) is an even Coxeter system with diagram V and V' is another diagram for W with odd edge $[xy]$. There exists a vertex $u \in V' - \{x, y\}$ such that u is contained in the intersection of all simplices σ' containing $\{x, y\}$ and such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for σ a simplex of V . Furthermore if σ' is a simplex of V' containing u and such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for σ a simplex of V , then $\{x, y\} \subset \sigma'$.*

Proof: Assume V' is a minimal (with respect to $T(V')$) counterexample. Let δ' be the intersection of all simplices σ' of V' , containing $\{x, y\}$ and such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for some simplex σ of V . As $\langle x, y \rangle$ is not an even Coxeter group, $\delta' \neq \{x, y\}$. If $u \in \delta' - \{x, y\}$, there is no odd path in V' from u to x or y , by Lemmas 9 and 10. For each $u \in \delta' - \{x, y\}$ assume there is a simplex β' of V' such that $\langle \beta' \rangle$ is conjugate to $\langle \beta \rangle$ for β a simplex of V and $u \in \beta'$, but $\{x, y\} \not\subset \beta'$. By intersecting, we may assume that each such $\beta' \subset \delta'$. Select one such β' . By Lemma 29 the map of $\langle \beta' \rangle$ to \mathbb{Z}_2 that sends $\beta' - \{x, y\}$ to 1 and $\beta' \cap \{x, y\}$ to -1 defines a smaller counterexample. •

Let δ' be the intersection of all simplices σ' of V' such that $\{x, y\} \subset \sigma'$ and $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for some simplex σ of V . By Lemma 30, δ' contains a vertex v such that if $v \in \sigma'$, where σ' is a simplex of V' and $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for some simplex σ of V , then $\{x, y\} \subset \sigma'$. We call such a vertex $\{x, y\}$ -linked or simply *linked*.

It suffices to show that δ' contains a linked vertex v' , such that every edge of V' containing v' is labeled 2. Otherwise, assume that V' is a minimal counterexample. Then each linked vertex belongs to an edge with label > 2 . Suppose $[st]$ is an edge of V' with label > 2 and neither s nor t is linked. If $[st]$ has even (respectively odd) label, then the even Coxeter group $W/N((st)^2)$ (respectively $W/N(st)$), with diagram \bar{V}' , obtained from V' by changing the label of $[st]$ to a 2 (respectively identifying s and t), is a smaller counterexample. (Note that if a vertex is not $\{x, y\}$ -linked in V' , then it is not $\{x, y\}$ -linked in \bar{V}' , and an $\{x, y\}$ -linked vertex of V' may not be $\{x, y\}$ -linked in \bar{V}' .) Hence, every edge with label > 2 (other than $[xy]$) contains a linked vertex.

Now $\langle \delta' \rangle$ is conjugate to $\langle \delta \rangle$ for some simplex δ of V . Then δ' contains more vertices than δ . (If the odd edges of δ' are collapsed to single vertices and each even > 2 label of V' is changed to 2, we obtain the (unique) diagram for a right angled (all edge labels are 2) Coxeter group. The diagram for this group is also obtained from δ if each even > 2 label of δ is changed to 2. This

latter description of this diagram has the same number of vertices as δ , but the former diagram has fewer vertices (from the collapse of $[xy]$) than δ' .)

We obtain the desired contradiction by showing δ has at least as many vertices as δ' . Let A' (A) be the set of vertices of δ' (δ) that belong to an edge of V' (V) with label > 2 . As $\langle \delta' \rangle$ is finite, no two adjacent edges of δ' have labels > 2 . Similarly for δ . The matching of Proposition 7 for δ' and δ respects the matching for V' and V . Hence δ' contains an edge with label > 2 iff δ contains the matching edge. So the number of vertices of A' that belong to an edge of δ' with label > 2 agrees with the number of vertices of A that belong to an edge of δ with label > 2 .

Suppose $[st]$ is an edge of V' with label > 2 and $s \in \delta'$, $t \notin \delta'$ and s is not a vertex of an edge in δ with label > 2 . Suppose $[ab]$ is the edge of V matching $[st]$. Then $\{a, b\} \not\subset \delta$. Considering the quotient of W by $N(\delta) = N(\delta')$, we see $\{a, b\} \cap \delta \neq \emptyset$. Hence we assume $b \in \delta$ and $a \notin \delta$. We wish to see that b does not belong to an edge of δ with label > 2 . Suppose $[bc]$ is such an edge and $[uv]$ is an edge of δ' matching $[bc]$. By Proposition 19 there is an odd edge path from $[st]$ to $[uv]$. The first edge of this path cannot be $[tp]$, since then $p \in \delta'$ and Lemma 9 is violated. Hence the first edge must be $[sp]$ and by assumption, $p \notin \delta'$, so $p \notin \{u, v\}$. If $[pq]$ is the next edge, then $q \in \delta'$ and again Lemma 9 is violated. We conclude that b does not belong to an edge of δ with label > 2 .

Recall from Proposition 13 that if edges with label > 2 of V' are adjacent, then their matching edges in V are adjacent. We show that $|A'| = |A|$ by verifying the following three statements.

First, if $[st]$ and $[uv]$ are (non-adjacent) edges with labels > 2 of V' such that $\{s, u\} \subset \delta'$, $t, v \notin \delta'$, and neither s nor u belongs to an edge of δ with label > 2 , then the corresponding edges of V , call them $[ab]$ and $[cd]$ respectively, are not adjacent. Otherwise, there is an odd edge path from $[st]$ to $[uv]$. A contradiction is obtained as in the former argument.

Suppose $[st]$ and $[sp]$ are edges of V' with labels > 2 , $s \in \delta'$, $\{t, p\} \subset V' - \delta'$ and s not a vertex of an edge of δ' with label > 2 . Then if $[ab]$ and $[ac]$ are the edges of V corresponding to $[st]$ and $[sp]$ respectively, either $a \in \delta$ and a is not adjacent to an edge of δ with label > 2 or $\{b, c\} \subset \delta$ and neither b nor c is adjacent to an edge of δ with label > 2 . We show the latter scenario cannot occur. Otherwise, the triangle $[abc]$ is contained in a maximal simplex τ of V , and $\langle \tau \rangle$ is conjugate to $\langle \tau' \rangle$ for τ' a maximal simplex of V' . Hence $\{s, t, p\}$ forms a triangle, no edge of which has odd label. By Lemmas 9 and 10, there is no odd edge path from $\{t, p\}$ to δ' and no odd edge path between

t and p . Let K be the kernel of the map of $\langle \sigma' \rangle$ to \mathbb{Z}_2 that takes $\sigma' - \{s\}$ to 1 and s to -1 . A diagram for $\bar{W} \equiv W/N(K)$ is obtained from V' by removing the vertices $\sigma' - \{s\}$ and all vertices that can be connected to $\sigma' - \{s\}$ by an odd labeled edge path. The subgroup $\langle s, t, p \rangle$ of W injects under this quotient map. By Lemma 29, a diagram for $W/N(K)$ is obtained from V by removing some vertices of δ and identifying all others. Neither b nor c are removed since Proposition 7 (applied to $W/N(K)$ and the two diagrams for this group) implies that the edges $[st]$ and $[sp]$ correspond to $[ab]$ and $[ac]$ respectively. Similarly b and c are not identified.

Finally, suppose $[st]$ and $[sp]$ are edges of V' with labels > 2 , $\{t, p\} \subset \delta'$, $s \in V' - \delta'$ and neither t nor p a vertex of an edge of δ' with label > 2 . Then if $[ab]$ and $[ac]$ are the edges of V corresponding to $[st]$ and $[sp]$ respectively, either $a \in \delta$ and a is not adjacent to an edge of δ with label > 2 or $\{b, c\} \subset \delta$ and neither b nor c is adjacent to an edge of δ with label > 2 . We show the former scenario cannot occur. As $\{s, t, p\}$ forms a triangle, no edge of this triangle has odd label. If each odd labeled edge of V' is identified to a vertex then the resulting diagram is even with Coxeter group \bar{W} a quotient of W . Another diagram for \bar{W} is obtained from V by changing labels on edges corresponding to odd labeled edges to 2. The triangles $[stp]$ and $[abc]$ induce triangles in the respective diagrams for \bar{W} and the conjugate simplices groups $\langle \sigma' \rangle$ and $\langle \sigma \rangle$ induce conjugate simplex groups in \bar{W} . Since both of these diagrams are even, the previous argument shows this is impossible.

By a completely analogous argument, we have:

Lemma 31 *If $\sigma' \subset \delta'$ is such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for $\sigma \subset \delta$, then $|A' \cap \sigma'| = |A \cap \sigma|$. •*

Let B' (B) be the vertices of δ' (δ) that do not belong to an edge with label > 2 . So $\delta' - A' = B'$ and $\delta - A = B$. It suffices to show $|B'| \leq |B|$. If $b' \in B'$, then there exists a simplex $\sigma'_{b'} \subset \delta'$ such that $b' \in \sigma'_{b'}$, $\langle \sigma'_{b'} \rangle$ is conjugate to $\langle \sigma_{b'} \rangle$ for some simplex $\sigma_{b'} \subset \delta$ and not both x and y are in $\sigma'_{b'}$. Since $\sigma'_{b'}$ can contain no linked vertex, it must be right angled and so $|\sigma'_{b'}| = |\sigma_{b'}|$. By Lemma 31, $|\sigma'_{b'} \cap A'| = |\sigma_{b'} \cap A|$, so $|\sigma'_{b'} \cap B'| = |\sigma_{b'} \cap B|$.

Lemma 32 *If $\sigma'_1, \dots, \sigma'_n$ are subsets of δ' , and $\langle \sigma'_i \rangle$ is right angled and conjugate to $\langle \sigma_i \rangle$ for $\sigma_i \subset \delta$ then $|B' \cap (\cap_{i=1}^n \sigma'_i)| = |B \cap (\cap_{i=1}^n \sigma_i)|$.*

Proof: Since V is an even diagram, $\langle \cap_{i=1}^n \sigma'_i \rangle$ is conjugate to $\langle \sigma \rangle$ for $\sigma \subset \cap_{i=1}^n \sigma_i$. Hence $|B' \cap (\cap_{i=1}^n \sigma'_i)| \leq |B \cap (\cap_{i=1}^n \sigma_i)|$, and it remains to show

the reverse inequality. We present the case $n = 2$. The general case is completely analogous. Assume $\langle \sigma_1 \cap \sigma_2 \rangle$ is conjugate to $\langle \bar{\sigma}'_1 \rangle$ for $\bar{\sigma}'_1 \subset \sigma'_1$, and also conjugate to $\langle \bar{\sigma}'_2 \rangle$ for $\bar{\sigma}'_2 \subset \sigma'_2$. As $\langle \bar{\sigma}'_1 \rangle$ is conjugate to $\langle \bar{\sigma}'_2 \rangle$, if $v \in \bar{\sigma}'_1$ then there is an odd edge path from v to some vertex of $\bar{\sigma}'_2$. But if $v \in B'$, it belongs only to edges labeled 2. Hence $\bar{\sigma}'_1 \cap B' = \bar{\sigma}'_2 \cap B' = \bar{\sigma}'_1 \cap \bar{\sigma}'_2 \cap B'$. Also, $|\sigma_1 \cap \sigma_2 \cap B| = |\bar{\sigma}'_i \cap B'| = |\bar{\sigma}'_1 \cap \bar{\sigma}'_2 \cap B'| \leq |\sigma'_1 \cap \sigma'_2 \cap B'|$. •

The sets $\sigma'_{b'} \cap B'$ for $b' \in \delta' \cap B'$ cover B' . (although it is not clear if the sets $\sigma_{b'}$ cover B). Lemma 32 and the Inclusion-Exclusion Principle imply $|B'| \leq |B|$ and the proof of Proposition 5 is complete. •

6 Loops of Size 4

Proposition 33 *Suppose (W, S) is an even Coxeter system with diagram V and V' is another diagram for W . If $[xy]$, $[yz]$, $[zv]$ and $[vx]$ are distinct edges of V' , and $[xy]$ and $[yz]$ have odd labels then there is an edge (labeled 2) between v and y .*

Proof: The edges $[vx]$ and $[zv]$ have label 2, by Lemmas 9 and 10. By Lemma 9 there is no edge $[xz]$. Suppose V' is a minimal counterexample (with respect to $T(V')$). We prove a collection of lemmas.

Lemma 34

- (1) *Every even label of an edge of V' is 2.*
- (2) *If $u \neq y$ is adjacent to x (resp. z) then $[ux]$ (resp. $[uz]$) has label 2.*
- (3) *Every odd labeled edge of V' is adjacent to y or v .*
- (4) *If $[yu]$ (resp. $[vu]$) has odd label then $(uv)^2 = 1$ (resp. $(uy)^2 = 1$).*

Proof: Otherwise a quotient map leads to a smaller counterexample. •

Let \bar{V} be the full subcomplex of V' with vertex set $\{v\}$ union all vertices of the odd labeled edges. (In particular, $\{x, y, z, v\} \subset \bar{V}$.)

Lemma 35 *Suppose $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for σ' and σ non-trivial simplices of V' and V respectively.*

- (1) *The simplex σ' contains a vertex of \bar{V} .*
- (2) *If σ' contains a vertex of $V' - \bar{V}$ then σ' contains two vertices of \bar{V} .*

Proof: If σ' contains no vertex of \bar{V} then $W/N(\sigma')$ is a smaller counterexample. If σ' contains a vertex of $V' - \bar{V}$ and t is the only vertex of $\sigma' \cap \bar{V}$, then by Lemma 29 $W/N(\sigma' - \{t\})$ is a smaller counterexample. •

Lemma 36 *Suppose $[x'y]$ is an odd labeled edge of V' (so $(x'v)$ has order 2). Then there is a vertex $t \in \bar{V}$ such that (tv) has odd order, and for every simplex σ' of V' containing $\{x', v\}$ and such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for σ a simplex of V , $t \in \sigma'$. (By Lemma 9, σ' cannot contain a vertex $t' \neq t$ such that $(t'v)$ has odd order. In this sense, t is unique.)*

Proof: Suppose σ' is a simplex of V' containing $\{x', v\}$ and such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for some simplex σ of V , then $x, y, z \notin \sigma'$. There is no vertex $t \in \sigma'$ such that $[yt]$ is an odd labeled edge by Lemma 9. If there is no vertex $t \in \sigma'$ such that (tv) has odd order, then (by Lemma 29) $W/N(\{x'v\} \cup (\sigma' - \{x', v\}))$ is an even Coxeter group and the diagram for this Coxeter group obtained from V' by identifying x' and v , and removing the vertices of $\sigma' - \{x', v\}$, contains a triangle that violates Lemma 9.

Now suppose that σ'_1 and σ'_2 are simplices of V' as above, and t_i is a vertex of σ'_i such that $t_i v$ has odd order and $t_1 \neq t_2$. Then $\{x', v\} \subset \sigma'_1 \cap \sigma'_2 \equiv \sigma'$. But then there is a $t_3 \in \sigma'$ such that $(t_3 v)$ has odd order. As t_3 is in both σ'_1 and σ'_2 , we have a contradiction to Lemma 9. •

Note that in the previous lemma ty has order 2.

By a completely analogous argument we have:

Lemma 37 *Suppose $[vt']$ is an odd labeled edge of V' (so $(t'y)$ has order 2). Then there is a vertex $x' \in \bar{V}$ such that $(x'y)$ has odd order, and for every simplex every simplex σ' of V' containing $\{t', y\}$ and such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for σ a simplex of V , $x' \in \sigma'$. (By Lemma 9, σ' cannot contain a vertex $x'' \neq x'$ such that $(x''y)$ has odd order. In this sense, x' is unique.) •*

Let $x \equiv x_1, x_2, \dots, x_n = z$ be the vertices of \bar{V} such that $(x_i y)$ has odd order. Let t_i be the vertex of Lemma 36 for $\{x_i, v\}$, so that $(t_i v)$ has odd order and $(t_i x_i)$ has order 2.

Next we show that $t_i \neq t_j$ for $i \neq j$. Otherwise, consider simplicies σ'_1 and σ'_2 containing triangles $[t_i x_i y]$ and $[t_i x_j y]$ respectively, such that $\langle \sigma'_i \rangle$ is conjugate to $\langle \sigma_i \rangle$ for some simplex σ_i of V . But then Lemma 37 implies $x_i = x_j$, which is nonsense.

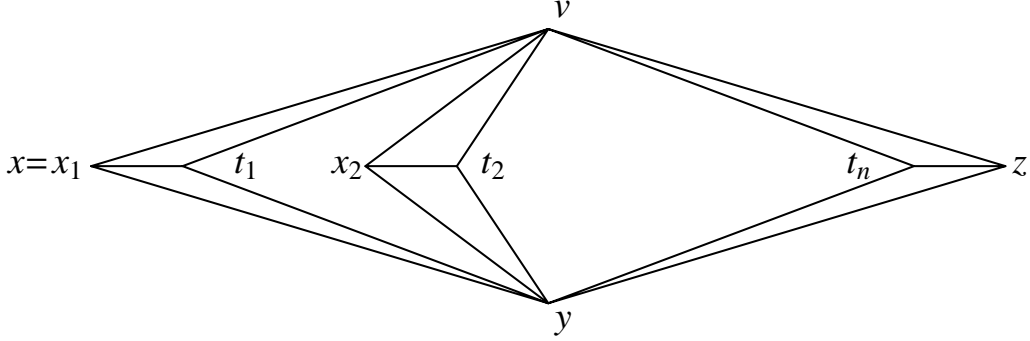


Figure 4:

Similarly there is no edge between t_i and x_j for $i \neq j$ (see Figure 4). Hence we have:

Lemma 38 *The only edges of \bar{V} are $[vx_i]$, $[vt_i]$, $[yx_i]$, $[yt_i]$ and $[x_it_i]$. •*

Lemma 39 *If s is a vertex of $V' - \bar{V}$, then there is no pair of edges e_1, e_2 such that e_1 connects s to a point of $\{x_i, t_i\}$ and e_2 connects s to a point of $\{x_j, t_j\}$ for $i \neq j$.*

Proof: Otherwise let σ'_1 and σ'_2 be maximal simplices containing e_1 and e_2 respectively. Now, $s \in \sigma'_1 \cap \sigma'_2 \equiv \sigma'$. By Lemma 35, σ' contains two vertices of \bar{V} and so t_l or x_l is an element of σ' for some l . But this is impossible by Lemma 38 •

Lemma 40 *The set $\{v, y\}$ separates x from z .*

Proof: Suppose there is an edge path $[xs_1], [s_1s_2], \dots, [s_mz]$ that does not intersect $\{v, y\}$. By Lemma 39, there is a smallest integer $1 < i \leq m$ such that s_i is not connected to x or t_1 by an edge. Note that $s_{i-1} \notin \bar{V}$ and s_{i-1} is connected to x or t_1 by an edge. Also, by Lemma 39 (with s_{i-1} in place of s) $s_i \notin \bar{V}$. Let σ' be a maximal simplex of V' containing $\{s_{i-1}, s_i\}$. By Lemma 35 there is $j \neq 1$ and vertex $u \in \{x_j, t_j\} \cap \sigma'$. But this is impossible by Lemma 39 applied to s_{i-1} . •

By Proposition 13 and Lemma 23 the odd labeled edges of V' containing v correspond to edges of V with common vertex a and the odd labeled edges

of V' containing y correspond to edges of V with common vertex b . The subcomplex of V composed of the edges with label > 2 and adjacent to a , and the subcomplex of V' composed of the edges with label > 2 and adjacent to b have trivial intersection by Proposition 19.

Let σ'_1 be a maximal simplex of V' containing the triangle $[x, t_1, v]$ and let σ'_2 be a maximal simplex of V' containing the triangle $[x, t_1, y]$. Let σ_1 and σ_2 be simplices of V such that $\langle \sigma'_i \rangle$ is conjugate to $\langle \sigma_i \rangle$. Now, σ_1 contains an edge $[ac]$ (with label > 2) corresponding to $[t_1v]$ and no vertex $c' \neq c$ such that $[ac']$ has label > 2 . Similarly, σ_2 contains an edge $[bd]$ corresponding to $[xy]$ and no vertex $d' \neq d$ such that $[bd']$ has label > 2 . By conjugation we assume that $\langle \sigma_1 \rangle = \langle \sigma'_1 \rangle$ and say $\langle \sigma'_2 \rangle = w\langle \sigma_2 \rangle w^{-1}$ for some $w \in W$. We have $\{x, t_1\} \subset \langle \sigma' \rangle \equiv \langle \sigma'_1 \rangle \cap \langle \sigma'_2 \rangle = \langle \sigma_1 \rangle \cap w\langle \sigma_2 \rangle w^{-1} = \langle \sigma \rangle$ for $\sigma \subset \sigma_1$. Since V is even, $\sigma \subset \sigma_2$. If $s \in \sigma$ is a vertex of an edge of V with label > 2 , then $s \in \{a, b, c, d\}$. Since σ' (and hence σ) is right angled, a or c is not in σ , and b or d is not in σ . Since $W/N(\sigma') = W/N(\sigma)$ is right angled a and b are elements of σ . We now have that $\langle a, b \rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

In a completely analogous manner we find a simplex τ' of V' containing $\{z, t_n\}$ such that $\langle \tau' \rangle$ is conjugate to $\langle \tau \rangle$ where τ is a simplex of V containing $\{a, b\}$. Let C be the component of $V' - \{v, y\}$ containing $\{x, t_1\}$ and $D = \Lambda' - C$ (so $\{z, t_n\} \subset D$). W decomposes as the amalgamated product $\langle C \cup \{v, y\} \rangle *_{\langle v, y \rangle} \langle D \rangle$. Since $\langle x, t_1 \rangle \sim \mathbb{Z}_2 \times \mathbb{Z}_2$ is not a subgroup of a conjugate of the edge group $\langle v, y \rangle \sim \mathbb{Z}_2 * \mathbb{Z}_2$, $\langle x, t_1 \rangle$ can not stabilize a vertex of T (the Bass-Serre tree for $\langle C \cup \{v, y\} \rangle *_{\langle v, y \rangle} \langle D \rangle$), other than $\langle C \cup \{v, y\} \rangle$. Since $\{x, t_1\} \subset \sigma$ and since $\langle \sigma \rangle$ stabilizes some vertex of T, $\langle \sigma \rangle$ stabilizes $\langle C \cup \{v, y\} \rangle$. Similarly, $\langle z, t_n \rangle$ only stabilizes the vertex $\langle D \rangle$ of T. As some conjugate $w\langle \tau \rangle w^{-1}$ contains $\langle z, t_n \rangle$, $w\langle \tau \rangle w^{-1}$ stabilizes $\langle D \rangle$ (equivalently $w\langle \tau \rangle w^{-1} < \langle D \rangle$). Hence, $\langle \tau \rangle < w^{-1}\langle D \rangle w$ and so $\langle \tau \rangle$ stabilizes the vertex $w^{-1}\langle D \rangle$ of T. But then $\langle a, b \rangle \sim \mathbb{Z}_2 \times \mathbb{Z}_2$ stabilizes distinct vertices of T. This is impossible as $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not a subgroup of $\mathbb{Z}_2 * \mathbb{Z}_2$. •

Proposition 41 *Suppose W is a finitely generated Coxeter group with even diagram V and diagram V' which is not even. If in V' , $[xy]$ has odd label, $[xu]$ and $[yz]$ have even labels, and $[uz]$ has odd label, then there is an (diagonal) edge $[xz]$ or $[yu]$.*

Proof: Otherwise assume V' is a minimal counterexample to the Proposition. Then we may assume that every even edge of V' is labeled 2 and by Proposition 33 and Lemma 10 there are no odd labeled edges, other than

$[xy]$ and $[uz]$. Let σ' be a maximal simplex of V' containing $\{u, x\}$. Then $z, y \notin \sigma'$. By Lemma 29, $W/N(\{ux\} \cup (\sigma' - \{u, x\}))$ is an even Coxeter group with diagram obtained from V' by identifying u and x , and removing the vertices of $\sigma' - \{u, x\}$. This contradicts Lemma 9. •

Lemma 42 *Suppose $\mathbb{Z}_2^n = \langle a, b, s_3, \dots, s_n \rangle = \langle a, b, e_3, \dots, e_n \rangle \equiv G$. Then there is a retraction $h : G \rightarrow \langle a, b \rangle$ such that $h(a) = a$, $h(b) = b$, $h(e_i) \in \{a, b\}$, and $h(s_i) \in \{a, 1\}$ for all i .*

Proof: Let $M_{n,n}$ be the coefficient matrix for s_i . I.e. $s_i = m_{i,1}a + m_{i,2}b + m_{i,3}e_3 + \dots + m_{i,n}e_n$, where each $m_{i,j} \in \{0, 1\}$.

The matrix M is invertible. As the block $[m_{i,j}]$ for $j > 2$ and $i > 2$ is invertible, a sequence of elementary row operations (only involving rows 3 through n) can be used to transform M to the matrix \bar{M} where $\bar{m}_{i,i} = 1$ for all i , $\bar{m}_{i,j} = 0$ for $i < j$ and for $i > j > 2$.

We now define h . If $\bar{m}_{i,1} = 1$ and $\bar{m}_{i,2} = 1$ then $h(e_i) = b$.

If $\bar{m}_{i,1} = 1$ and $\bar{m}_{i,2} = 0$ then $h(e_i) = a$.

If $\bar{m}_{i,1} = 0$ and $\bar{m}_{i,2} = 1$ then $h(e_i) = b$.

If $\bar{m}_{i,1} = 0$ and $\bar{m}_{i,2} = 0$ then $h(e_i) = a$.

To understand the effect this has on $h(s_i)$, some notation is helpful. In \bar{M} , if $\bar{m}_{i,1} = 1$, then replace it by 1_a . If $\bar{m}_{i,2} = 1$, then replace it by 1_b . If $h(e_i) = a$ (respectively b) replace $\bar{m}_{i,i}$ by 1_a (respectively 1_b). In each row of \bar{M} (except the second) there are an even number of 1_b -entries. Reversing the above row operations to obtain M from \bar{M} we see that if $\bar{m}_{j,j} = 1_a$ (respectively 1_b) then $m_{i,j} \in \{0, 1_a\}$, (respectively $m_{i,j} \in \{0, 1_b\}$) and $h(e_j) = a$ (respectively $h(e_j) = b$). Furthermore, each row of M (except the second) contains an even number of 1_b -entries. I.e. $h(s_i) \in \{1, a\}$. •

Proposition 43 *Suppose W is a finitely generated Coxeter group with even diagram V and diagram V' which is not even. If $[xy]$ has odd label in V' and $(xyst)$ is a simple loop in V' , then there is an (diagonal) edge $[xs]$ or $[yt]$.*

Proof: Assume V' is a minimal counterexample to the proposition. By Propositions 33 and 41, $[xy]$ is the only odd labeled edge of $(xyst)$. Each even edge of V' is labeled 2. By Lemma 10, any odd labeled edge not containing x or y must contain a vertex of $\{s, t\}$ and this edge must be connected to the diagonally opposite vertex of $\{x, y\}$ by an edge labeled 2. By Proposition 33, there can be at most one odd labeled edge containing a given vertex of $\{x, y, s, t\}$.

Case 1. The only odd edge of V' is $[xy]$.

By Lemma 29 every simplex σ' of V' such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for σ a simplex of V is either $\{u\}$ for $u \in \{x, y, s, t\}$ or contains two vertices of $\{x, y, s, t\}$, but not three. By Lemma 30 there is a vertex $u \in V'$ such that every simplex σ' of V' containing u and such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for σ a simplex of V contains $\{x, y\}$, and every simplex τ' of V' containing $\{x, y\}$ such that $\langle \tau' \rangle$ is conjugate to $\langle \tau \rangle$ for τ a simplex of V contains u .

We show that $\{x, y\}$ separates u from t . Choose a shortest edge path avoiding $\{x, y\}$ from u to t . Let v be the first vertex of this path that is not adjacent to both x and y and let v' be the previous vertex. Choose a maximal simplex σ'_1 containing $\{x, y, v'\}$ and a maximal simplex σ'_2 containing $\{v, v'\}$. Let $\sigma' = \sigma'_1 \cap \sigma'_2$. Assume σ'_2 does not contain x . Then σ' does not contain x, s or t , which is impossible.

Now as $\{x, y\}$ separates V' , [6] implies that W visually decomposes (with respect to V') as $\langle A \rangle *_{\langle x, y \rangle} \langle B \rangle$, where $A \cup B$ is the vertex set of V' and A and B properly contain $\{x, y\}$. By [6], there is a visual (with respect to V) decomposition $C *_E D$ of W such that E is a subgroup of a conjugate of $\langle x, y \rangle$. This implies that E is either trivial or $\langle v \rangle$ for some vertex v of V . Then W visually decomposes (with respect to V') as $\langle F \rangle *_{\langle H \rangle} \langle G \rangle$ where H is trivial or $\{v'\}$ for some vertex v' of V' , $F \cup G$ is the vertex set of V' and H is a proper subset of F and G . As $\langle x, y, s, t \rangle$ is 1-ended, we must have $\{x, y, s, t\}$ a subset of F or G . Assume $\{x, y, s, t\} \subset F$. Let e be a vertex of $G - H$ and σ' be a maximal simplex of V' containing e . Then $\sigma' \cap \{x, y, s, t\}$ is either \emptyset or v' . This is impossible as σ' must contain two vertices of $\{x, y, s, t\}$. •

Hence there must be an odd edge at s or t .

Case 2. Assume there is an odd edge $[tv]$ but no odd edge at s .

There is an edge $[yv]$ labeled 2 by Lemma 10 and an edge $[vx]$ labeled 2 by Proposition 41. There are no odd labeled edges, other than $[xy]$ and $[vt]$. Every simplex σ' (with at least two vertices) such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for some simplex σ of V must contain at least two vertices of $\{x, y, s, t, v\}$. There may or may not be an edge $[sv]$ labeled 2. By Lemma 29 there is a vertex u of V' such that if σ' is a simplex of V' , $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for some simplex σ of V and $x, y \subset \sigma'$ then $u \in \sigma'$. Also if α' is a simplex of V' , $\langle \alpha' \rangle$ is conjugate to $\langle \alpha \rangle$ for some simplex α of V and $u \in \alpha'$ then $x, y \in \alpha'$. Note that $u \neq v$.

Let $[ab]$ and $[cd]$ be the edges of V corresponding to (see Proposition 7) $[xy]$ and $[tv]$ respectively. Note that $\{a, b\} \cap \{c, d\} = \emptyset$ by Proposition 19.

Lemma 44 $\{x, y, v\}$ separates u from t (and s).

Proof: Otherwise, there are consecutive vertices $u = u_0, \dots, u_n = t$ such that no $u_i \in \{x, y, v\}$. Assume that i is the first integer such that u_i does not commute with x and y . Suppose u_i does not commute with x (the case u_i does not commute with y is completely analogous). Let $\sigma' = \sigma'_1 \cap \sigma'_2$ where σ'_1 is a maximal simplex containing $\{x, y, u_{i-1}\}$ and σ'_2 is a maximal simplex containing $\{u_{i-1}, u_i\}$. Then $t, s, x \notin \sigma'$ and so $u_{i-1}, y, v \in \sigma'$.

Observe that $\sigma' - \{y\} \subset lk_2(x, y)$.

Suppose σ is a simplex in V such that $\langle \sigma \rangle$ is conjugate to $\langle \sigma' \rangle$. By conjugation, we may assume that $\langle \sigma \rangle = \langle \sigma' \rangle$. We may assume that $a, c \in \sigma$ and $b, d \notin \sigma$, since $\langle \sigma \rangle$ is abelian. Let $N = N(\sigma - \{a, c\})$. Note that $y, v \notin N$.

Now we show:

(*) If $p \in \langle \sigma' - \{y\} \rangle$, then $yp \notin N$.

Suppose $yp \in N$. As yp conjugates $xy (= w(ab)^2 w^{-1})$ to yx , $w^{-1}ypw$ conjugates $(ab)^2$ to $(ba)^2$. Proposition 12 implies that $w^{-1}ypw = ef$ for $e \in lk_2(a, b)$ and f of odd length in $\langle a, b \rangle$. As $yp \in \langle \sigma \rangle$, $yp = ag$ for $g \in \langle \sigma - \{a\} \rangle$. (Else, $yp = g \in \langle \sigma - \{a\} \rangle$ implying $ef = w^{-1}gw$ implying (the odd length element of $\langle a, b \rangle$) $f = ew^{-1}gw$. But $ew^{-1}gw \in N(S - \{a, b\})$ so this is impossible.) But $ag \notin N$ and (*) is proved.

Let $q : W \rightarrow W/N$ be the quotient map. Note that $q\langle \sigma' \rangle = q\langle \sigma \rangle = \langle q(a), q(c) \rangle \cong Z_2 \times Z_2$. As y, v and yv are not elements of N , $q\langle \sigma' \rangle = \langle q(y), q(v) \rangle$. To finish the Lemma, we show that W/N is a smaller counterexample.

First we show that if $m \in \sigma' - \{y, v\}$ then $q(m) \in \{1, q(v)\}$: If $q(m) \neq 1$, then $q(m) \in \{q(y), q(v), q(yv)\}$. If $q(m) = q(y)$ then $q(my) = 1$ implying $my \in N$, contrary to (*). If $q(m) = q(yv)$, then $q(mvy) = 1$ implying $mvy \in N$, contrary to (*). Hence $q(m) = q(v)$.

If K is the kernel of the restriction of q to $\langle \sigma' \rangle$, then the normal closure of K in W is N . As K is generated by $\{m \in \sigma' : q(m) = 1\} \cup \{mv : m \in \sigma' \text{ and } q(m) = q(v)\}$, a diagram for the even Coxeter group W/N is obtained from V' by removing the vertices of $(\sigma' - \{y, v\}) \cap \ker(q)$ and identifying the remaining vertices of $\sigma' - \{y, v\}$ with v . •

We can now finish Case 2. By [6] there is a simplex σ in V such that σ separates V and $\langle\sigma\rangle$ is a subgroup of a conjugate of $\langle x, y, v \rangle$. Note that the edge $[ab]$ corresponding to $[xy]$ is not in σ . (Since otherwise, the pigeon-hole principle implies $\sigma = \{a, b\}$ and $\langle a, b \rangle$ is conjugate to $\langle x, y, v \rangle$. This is impossible by since $\{v, t\} \subset N(x, y, z)$, but $\langle c, d \rangle$ injects under the quotient $W \rightarrow W/N(a, b)$.) Hence $\langle\sigma\rangle$ is abelian and is either isomorphic to \mathbb{Z}_2 or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Again applying [6], either an edge (labeled 2) or vertex of V' must separate V' and the group generated by the vertices of this separating set is a subgroup of a conjugate of $\langle x, y, v \rangle$. The set $\{x, y, s, t, v\}$ generates a 1-ended group and so no vertex w of V' can separate V' . (Otherwise, there is a component C of $V' - \{w\}$ such that $\{x, y, s, t, v\} \subset C \cup \{w\}$. But if k is a vertex of a component $K \neq C$ of $V' - \{w\}$, then a maximal simplex of V' containing k could contain at most one vertex of $\{x, y, s, t, v\}$, which is impossible.) If an edge labeled 2 separates V' and the group for this edge is conjugate to a subgroup of $\langle x, y, v \rangle$, then the only candidates are the edges $[xv]$, $[yv]$ and $[xt]$. The groups for these edges are all conjugate. All cases are completely similar, and we assume $[yv]$ separates V' . Then (by [6]) there is an edge separating V such that the corresponding group is conjugate to $\langle y, v \rangle$. We may assume that this edge is $[ac]$. By conjugation, we may assume that $\langle y, v \rangle = \langle a, c \rangle$. Note that vy conjugates $xy (= w_1(ab)^2w_1^{-1})$ to $yx (= w_1(ba)^2w_1^{-1})$.

If $vy = c$, then $cw_1(ab)^2w_1^{-1}c = w_1(ba)^2w_1^{-1}$. But in $W/N(\{c\})$, $(ab)^2 \neq (ba)^2$, so this is impossible. If $vy = a$, then choose $g \in \langle x, y \rangle$ such that $gvyg^{-1} = xv$. As xv conjugates $tv (= w_2(cd)^2w_2^{-1})$ to $vt (= w_2(dc)^2w_2^{-1})$, gag^{-1} conjugates $w_2(cd)^2w_2^{-1}$ to $w_2(dc)^2w_2^{-1}$. But in $W/N(\{a\})$, $(cd)^2 \neq (dc)^2$, so this is impossible. The only other possibility is $vy = ac$. Hence we have $\{y, v\} = \{a, c\}$.

The group $\langle x, y, s, t, v \rangle$ is 1-ended and hence $\{x, y, s, t, v\}$ is a subset of $\{v, y\}$ union a component C of $V' - \{v, y\}$. Let σ' be a maximal simplex containing a vertex of a component of $V' - \{v, y\}$ other than C . Then σ' must also contain $\{v, y\} (= \{a, c\})$ and each edge label of σ' is 2. Assume that $\langle\sigma'\rangle = w\langle\sigma\rangle w^{-1}$ for σ a simplex of V . Then $a, c \in \sigma$, $waw^{-1} = a$ and $wcw^{-1} = c$. We wish to apply Lemma 42 to obtain a smaller counterexample.

Write $\langle\sigma'\rangle = \langle a, c, e_3, \dots, e_n \rangle = \langle a, c, ws_3w^{-1}, \dots, ws_nw^{-1} \rangle = \langle w\sigma w^{-1} \rangle$ and suppose h is the retraction of this group to $\langle a, c \rangle$ defined in Lemma 42. If $h(ws_iw^{-1}) = a$, then observe that $ws_iw^{-1}a = w(s_ia)w^{-1}$. Thus, an even diagram for $W/N(\ker(h))$ is obtained from V by removing all s_i such that $h(ws_iw^{-1}) = 1$ and identifying s_i with a if $h(ws_iw^{-1}) = a$. Another diagram

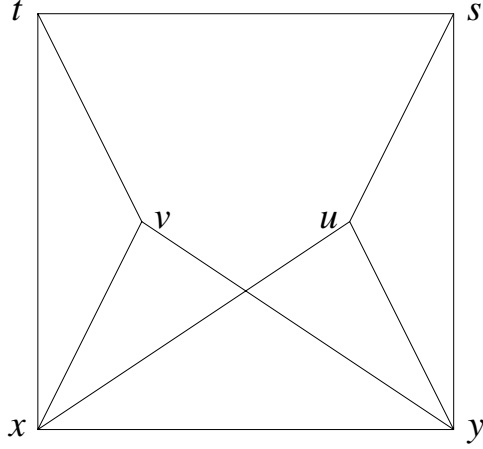


Figure 5:

for $W/N(\ker(h))$ is obtained from V' by identifying e_i with v when $h(e_i) = v$ and identifying e_j with y when $h(e_j) = y$. Then $W/N(\ker(h))$ is a smaller counterexample, finishing Case 2.

Now the final case.

Case 3. Assume there are edges $[tv]$ and $[su]$ with odd labels. Then there are edges $[uy]$ $[ux]$ $[vy]$ and $[vx]$ with labeled 2. (See Figure 5)

Let $\{a, b, c, d, e, f\} \subset V$ be vertices such that the edge correspondence of Proposition 7 relates $[xy]$ to $[ab]$, $[tv]$ to $[cd]$ and $[us]$ to $[ef]$. By Proposition 19, the edges $[ab]$, $[cd]$ and $[ef]$ are mutually disjoint. By Theorem 26 there is an edge $[gh] \in \{[ab], [cd], [ef]\}$ such that every other edge of V containing g is labeled 2 and such that if $[gk]$ is such an edge, then $[kh]$ is also an edge of V . We call g a *special vertex* of $[gh]$. A quotient argument shows that each of $[ab]$, $[cd]$, $[ef]$ contains a special vertex. E.g. if the edge labels on $[cd]$ and $[ef]$ are changed to 2, then Theorem 26 implies that $[ab]$ must have a special vertex. The next lemma implies that there cannot be an edge $[uv]$ in the minimal counterexample.

Lemma 45 *Suppose (W, S) is a finitely generated even Coxeter system, V' is a diagram for W with non-intersection odd edges $[xy]$, $[tv]$ and $[us]$, and even edges $[xu]$, $[xv]$, $[xt]$ $[yu]$, $[ys]$, $[yv]$, $[ts]$ and $[uv]$, (and so a tetrahedron $[xyuv]$ and triangles $[xtv]$ and $[suy]$). Then there is an edge (labeled 2) $[yt]$ or $[xs]$.*

Proof: If an edge not listed in the hypothesis, between two vertices of $\{x, y, u, v, t, s\}$ exists, it must have label 2 by Lemma 9. Assume V' is a minimal counterexample to the Lemma. All even edges of V' are labeled 2 by the minimality of V' . Every odd edge of V' contains a vertex of $\{x, y, u, v, t, s\}$ (otherwise collapse for a smaller counterexample). Suppose V' contained an odd edge other than $[xy]$, $[tv]$ or $[us]$. Then this edge must contain x, y, s or t (otherwise collapse). If $[xw]$ is an odd edge for $w \neq y$, there must be an edge $[ws]$ (or collapsing would give a smaller counterexample). In this situation, Proposition 33 implies there is an edge $[xs]$ and we are finished. Similarly if there is an odd edge at y, s or t . Hence we may assume there is no odd edge of V' other than $[xy]$, $[tv]$ and $[us]$.

We assume a, d and f are special vertices of $[ab]$, $[cd]$ and $[ef]$ respectively. Observe that if σ is a simplex of V containing a then $\sigma \cup \{b\}$ is a simplex of V . Similarly for d and f . Let σ'_1 be a maximal simplex of V' containing $\{x, y, v, u\}$. Then $\langle \sigma'_1 \rangle$ is conjugate to $\langle \sigma_1 \rangle$ for σ_1 a (maximal) simplex of V . Now $a, b \in \sigma_1$ and $c, e \in \sigma_1$ (as $\{a, b\}$ does not commute with $\{c, d\}$ or $\{e, f\}$). Similarly, considering the triangles $[yus]$ and $[tvx]$, we see that $\{b\}$ commutes with $\{e, f, c, d\}$.

Consider the loop $(tvus)$. By Proposition 41, there must be an edge $[tu]$ or $[vs]$. If both exist, then $\{c, d\}$ commutes with $\{e, f\}$ and there is a maximal simplex of V containing $\{b, c, d, e, f\}$, implying there is a maximal simplex in V' containing $\{t, v, u, s\}$ and either x or y , which is impossible in our minimal counterexample.

Now suppose $[vs]$ is an edge of V' , but $[tu]$ is not. (See Figure 6.)

Let σ'_2 be a maximal simplex of V' containing $\{u, s, y, v\}$. Then $\langle \sigma'_2 \rangle$ is conjugate to $\langle \sigma_2 \rangle$ for σ_2 a (maximal) simplex of V and $\{e, f, b, c\} \subset \sigma_2$. Choose a maximal simplex σ'_3 of V' containing $\{t, v, x\}$. Then there exists σ_3 , a maximal simplex of V containing $\{c, d, b\}$ and such that $\langle \sigma'_3 \rangle$ is conjugate to $\langle \sigma_3 \rangle$. Let σ'_4 be a maximal simplex of V' containing $\{t, v, s\}$. Then there exists σ_4 , a maximal simplex of V containing $\{c, d\}$ and either f or e , and such that $\langle \sigma'_4 \rangle$ is conjugate to $\langle \sigma_4 \rangle$. But then either $\{b, c, d, f\}$ or $\{b, c, d, e\}$ is a simplex. This implies there is a simplex σ' of V' containing $\{t, v\}$, a vertex of $\{x, y\}$ and a vertex of $\{u, s\}$. This is impossible and $[vs]$ is not an edge of V' .

Next assume that $[tu]$ is an edge, but $[sv]$ is not an edge of V' . (See Figure 7.)

We may assume that $\{x, y, u, v\} \subset \sigma'_1$ and $\{a, b, c, e\} \subset \sigma_1$ as above. Also, $\{x, u, v, t\} \subset \sigma'_2$ and $\{c, d, b, e\} \subset \sigma_2$; $\{u, s, y\} \subset \sigma'_3$ and $\{e, f, b\} \subset \sigma_3$; and

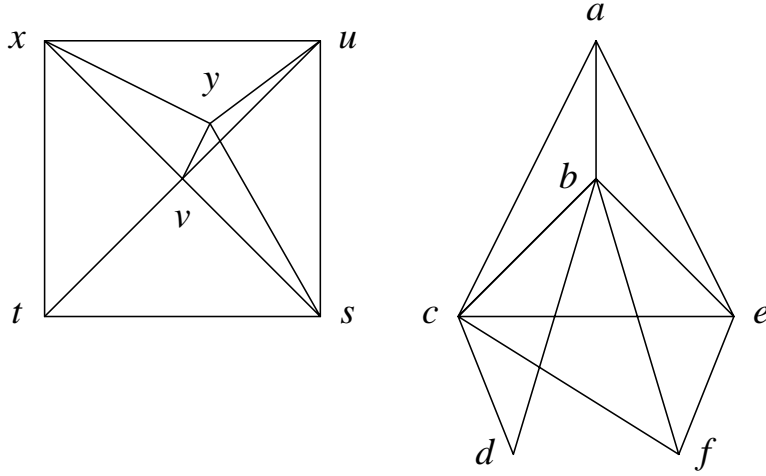


Figure 6:

$\{t, u, s\} \subset \sigma'_4$ and σ_4 contains $\{e, f\}$ and either c or d . But then either $\{e, f, c, b\}$ or $\{e, f, d, b\}$ is a simplex.

This implies there exists a simplex σ' of V' containing $\{u, s\}$, a vertex of $\{x, y\}$ and a vertex of $\{v, t\}$. This is impossible. •

Remark 4 Consider the loop $(suxt)$. There must be an (labeled 2) edge $[ut]$ or $[uv]$, or collapsing $[vt]$ gives a smaller counterexample. Consider the loop $(syvt)$. There must be an (labeled 2) edge $[vs]$ or $[uv]$, or collapsing $[us]$ gives a smaller counterexample. Either of these observations (along with Lemma 45) could be used in conjunction with the ideas of §7 to complete a proof of Case 3.

Lemma 45 implies $[uv]$ is not an edge of V' . At this point Figure 8 is our model.

Again if σ' is a simplex of V' such that $\langle \sigma' \rangle$ is conjugate to a $\langle \sigma \rangle$, for σ a simplex of V , and σ' contains a vertex of $V' - \{x, y, t, s, u, v\}$, then σ' must contain two vertices of $\{x, y, t, s, u, v\}$.

Lemma 46 *The set $\{x, y, u, v\}$ separates V' .*

Proof: Suppose not. By Lemma 30, there is a vertex z of $V' - \{x, y\}$ such that every simplex σ' of V' containing $[xy]$ and such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for σ a simplex of V , contains z and every simplex σ' of V' containing z

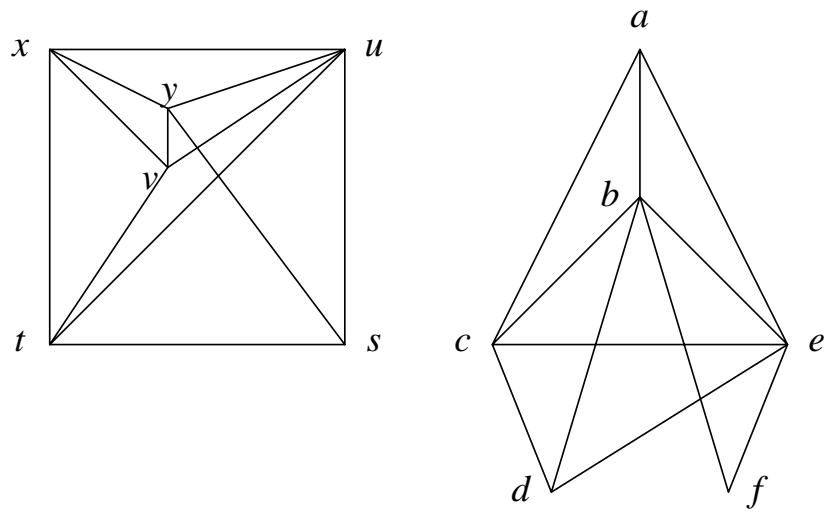


Figure 7:

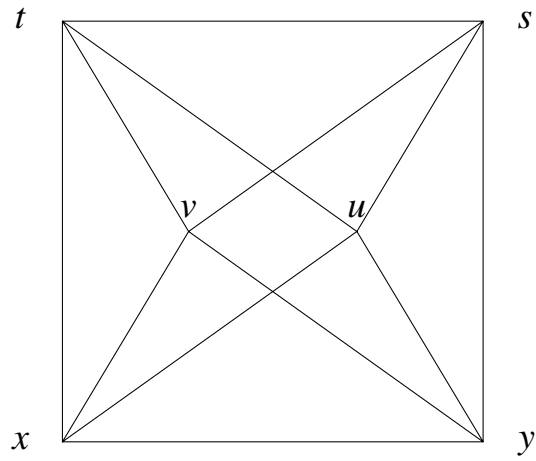


Figure 8:

such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for σ a simplex of V , contains $[xy]$. Clearly $z \notin \{v, u, t, s\}$. Suppose $z = z_0, z_1, \dots, z_n = t$ are consecutive vertices of $V' - \{x, y, u, v\}$. Assume that i is the first integer such that z_i is not adjacent to both x and y . Both cases are analogous, and we assume that z_i is not adjacent to y . Let σ'_1 be a maximal simplex of V' containing $\{x, y, z_{i-1}\}$, σ'_2 a maximal simplex containing $\{z_{i-1}, z_i\}$. Let $\sigma' = \sigma'_1 \cap \sigma'_2$. Then $\{y, s, t\} \cap \sigma' = \emptyset$. Then σ' must contain z_{i-1} and either $\{x, u\}$ or $\{x, v\}$ ($[vu]$ is not a possibility by Lemma 45). Both cases have analogous proofs and we assume $\{x, u\} \subset \sigma'$. Say σ is a simplex of V and $\langle \sigma \rangle$ is conjugate to $\langle \sigma' \rangle$. Then we may assume $b, e \in \sigma$ and $a, c, d, f \notin \sigma$. Let $N = N(\sigma - \{b, e\})$. Each vertex of $\sigma' - \{x\}$ commutes with x and y . An argument completely analogous to that for statement (*) in the proof of Lemma 44 implies that if $p \in \langle \sigma' - \{x\} \rangle$, then $xp \notin N$. Just as in the argument following the proof of (*), this implies W/N is a smaller counterexample. •

By [6] there is a full subgraph A separating V such that $\langle A \rangle$ is conjugate to a subgroup of $\langle u, v, x, y \rangle = \langle u, x, y \rangle *_{\langle x, y \rangle} \langle u, x, y \rangle$. The edge $[ab]$ is not in A , for otherwise, a conjugate of $\langle a, b \rangle$ is a subgroup of $\langle u, x, y \rangle$ or $\langle v, x, y \rangle$. But all three of these groups have the same order. This would imply $\langle a, b \rangle$ is conjugate to $\langle u, x, y \rangle$ or $\langle v, x, y \rangle$, but clearly $N(\{a, b\})$ is not equal to $N(\{u, x, y\})$ or $N(\{v, x, y\})$. So, A is right angled. The group $\langle x, y, u, v \rangle = \langle x, y, v \rangle *_{\langle x, y \rangle} \langle x, y, u \rangle$ is 2-ended and contains no copy of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. This implies that $\langle A \rangle = 1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 * \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 *_{\mathbb{Z}_2} \mathbb{Z}_2 \times \mathbb{Z}_2$. Using [6] again, a subset B of the vertices of V' separates V' such that $\langle B \rangle$ is right angled and a subgroup of a conjugate of $\langle A \rangle$ and so a subgroup of a conjugate of $\langle x, y, u, v \rangle$. This implies that B is a subset of $\{u, x, v\}, \{u, y, v\}, \{x, t\}$, or $\{y, s\}$. By Corollary 6 of [6] and we may assume that $\langle A \rangle$ is conjugate to $\langle B \rangle$. Note that $\{x, y, u, v, s, t\}$ is a subset of B union a component of the complement of B . Select a maximal simplex σ' intersecting another component of the complement of B . Then $\sigma' \cap B$ must be equal to $\{x, t\}, \{x, v\}, \{x, u\}, \{y, v\}, \{y, s\}$ or $\{y, u\}$ and be conjugate to a simplex of V . Now proceed as in Case 2. •

7 The Proof of Theorem 2

In this section we finish the proof of Theorem 2. Throughout this section, we assume that V' is a smallest diagram for an even Coxeter group such that V' contains a simple loop l without shortcuts, the length of l is ≥ 5 and l

contains an odd labeled edge $[xy]$. By the minimality of V' , all even edges of V' are labeled 2. By Lemma 10, Proposition 43 and the minimality of V' , all edges of l other than $[xy]$ are labeled 2.

Lemma 47 *If l' is a loop of V' containing 2 odd labeled edges, then l' must have a shortcut.*

Proof: Otherwise, the diagram obtained from V' by collapsing one of the edges of l' contradicts Proposition 43 or is a smaller example than V' . •

Lemma 48 *If $[xu]$ has odd label, then $u = y$.*

Proof: By Lemma 10, u is connected to a vertex of $l - st(x)$ by an edge. Say the consecutive vertices of l are $x = a_0, a_1, \dots, a_n = y$. Let i be the largest integer such that $[ua_i]$ is an edge of V' . By Lemma 9 and Proposition 33, $i < n - 1$. The loop with consecutive vertices $(yxa_i \dots a_{n-1})$ has no shortcuts, contradicting Lemma 47. •

Lemma 49 *If $[uv] \neq [xy]$ is an odd labeled edge, then $[uv]$ has one vertex on l and the other vertex in $lk_2(x, y)$.*

Proof: If neither u nor v is a vertex of l , then $W/N(uv)$ is a smaller example. Assume v is a vertex of l . If v is not adjacent to x or y (and $u \notin lk_2(x, y)$), then the quotient of W by $N(uv)$ gives a smaller example. If say v is adjacent to x , then u must be adjacent to y or again $W/N(uv)$ is a smaller example. Now by Lemma 47 (applied to the loop $(vuyx)$), u is adjacent to x . •

Lemma 50 *If $[uv]$ is an odd labeled edge of V' then $[uv]$ is contained in a simple loop of length ≥ 5 without shortcuts.*

Proof: Suppose otherwise. We assume that $v \in l$ and $u \in lk_2(x, y)$. If s, t are the vertices of l adjacent to v , then $s, t \in lk_2(u)$ or $[uv]$ belongs to a loop of length ≥ 5 without shortcuts (all edges of this path not containing u would be in l). Note that no vertex of $l - \{s, t\}$ belongs to $lk_2(v)$ (and so no such vertex belongs to $lk_2(u, v)$).

Next we show that if $[vw]$ is such that $w \neq u$ and $w \notin lk_2(u, v)$, then there is no edge path in $V' - lk_2(u, v)$ from w to a vertex of l . Otherwise, there is a simple edge path loop containing $[uv]$ and avoiding $lk_2(u, v)$. A shortest such loop contradicts the hypothesis on $[uv]$.

Now, let U be the union of all components K of $V' - lk_2(u, v)$ such that for some vertex $w \in K$, $[wv]$ is an edge. We have $U \cap l = \emptyset$. Twist U around $[uv]$ to form the diagram V'' for W . If $z \neq u$ and $[vz]$ is an edge of V'' , then $z \in lk_2(u, v)$. By Proposition 5 there is a vertex w ($\notin \{s, t\}$) of V'' such that the triangle $[uvw]$ can be replaced by an edge $[uz]$ with even label > 2 . The resulting diagram for W is smaller than V' and retains l (with v replaced by z). •

Lemma 51 *If v is a vertex of l not adjacent to x or y , then there is no odd edge at v .*

Proof: Assume that $[uv]$ is such an odd edge. By Lemma 49, $u \in lk_2(x, y)$. Let l' be a simple edge path without shortcuts containing $[uv]$ and having length ≥ 5 . Then x or y is a vertex of l' (otherwise, $W/N(xy)$ is a smaller example). We may assume x is a vertex of l' . By Lemma 49, $y \in lk_2(u, v)$. This is impossible as $[yv]$ would be a shortcut in l . •

Lemma 52 *If $s \neq y$ and $t \neq x$ are vertices of l adjacent to x and y respectively, then there is not an odd labeled edge at s and an odd labeled edge at t .*

Proof: Otherwise, say $[su]$ and $[tv]$ have odd labels. By Lemmas 50 and 48 $u \neq v$. By Lemma 49, $u, v \in lk_2(x, y)$. Assume l_s and l_t are simple loops without shortcuts and of length ≥ 5 containing $[su]$ and $[tv]$ respectively. By Lemma 49, l_s and l_t contain y and x respectively. Hence the paths (suy) and (tvx) are contained in l_s and l_t respectively. Since there is no edge from t to s , Lemma 49 implies that $v \in lk_2(s, u)$. By Lemma 49, l_t contains s or u and so the vertex of l_t following the path (tvx) must be s or u , but this is impossible as both are connected to v by an edge (creating a shortcut in l_t). •

We can now complete the reduction.

Case 1. Suppose the only odd labeled edge of V' is $[xy]$.

Then say $(xyst)$ is a subpath of l . Let σ' be a simplex of V' containing $\{s, t\}$ such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for some simplex σ of V . By Lemma 29, the quotient of W by $N(\{st\} \cup (\sigma' - \{s, t\}))$ is an even Coxeter group. A diagram for this group is obtained from V' by identifying s and t and removing the vertices of $\sigma' - \{s, t\}$. This diagram contains a loop of length

≥ 4 , with odd labeled edge $[xy]$ and no shortcuts. This diagram is smaller than V' , contradicting Proposition 43 or the minimality of V' .

Case 2. Suppose V' contains exactly two odd labeled edges.

Say $(uxyst)$ is a subpath of l and $[uv]$ is an odd labeled edge. By Lemma 49, $v \in lk_2(x, y)$. If $[cd]$ is an edge of l such that $\{c, d\} \cap \{u, x, y\} = \emptyset$, and σ' is a simplex of V' containing $[cd]$ such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for some simplex σ of V , then σ' must contain v (otherwise, the quotient of W by $N(\{cd\} \cup (\sigma' - \{c, d\}))$ gives a smaller example). In particular, v is connected to each vertex of l by an edge.

Let l' be a simple edge loop in V' of length ≥ 5 , containing $[uv]$, and without shortcuts. Then $l \cap l' = \{u, y\}$. Note that (uvy) is a subpath of l' . Let σ' be a simplex of V' containing $[stv]$ such that $\langle \sigma' \rangle$ is conjugate to $\langle \sigma \rangle$ for some simplex σ of V . Then $u, x, y \notin \sigma'$ and the only vertex of l' in σ' is v . A diagram for the even Coxeter group $W/N(\sigma' - \{v\})$ is obtained from V' by removing the vertices of $\sigma' - \{v\}$. But this diagram contains a faithful copy of l' and so contradicts the minimality of V' . The proof of Theorem 2 is complete.

References

- [1] P. Bahls, *Even Rigidity in Coxeter Groups*, Ph.D. Thesis, Vanderbilt University, 2002.
- [2] R. E. Borcherds *Coxeter Groups, Lorentzian Lattices and K3 Surfaces*, IMRN **19** (1998), 1011-1031.
- [3] B. Brink and R. Howlett, *Normalizers of Parabolic Subgroups in Coxeter Groups*, *Inventiones Mathematicae* **136** (1999), 323-351.
- [4] N. Bourbaki, *Groupes et Algebres de Lie* Chapters IV-VI, Hermann, Paris, 1968.
- [5] N. Brady, J. P. McCammond, B. Muhlherr and W. D. Neumann, *Rigidity of Coxeter Groups and Artin Groups*, *Geom. Dedicata* **94** (2002), 91–109.
- [6] M. Mihalik and S. Tschantz, *Visual Decompositions of Coxeter Groups*, Preprint.

- [7] J. Rotman, *An Introduction to the Theory of Groups*, 4th ed., Graduate Texts in Mathematics 148, Springer Verlag, 1995.