SPLITTING HOMOTOPY EQUIVALENCES ALONG TWO-SIDED SUBMANIFOLDS, II: RELATING THEORIES

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Abstract. In this paper, we relate two surgery theories of two-sided, codimension one submanifolds. First, we consider the Poincaré embedding obstruction groups LS of C.T.C. Wall [Wal99, §11]. The advantage is that no ambient surgeries are required in defining the obstruction. Next, we consider the splitting obstruction groups UNil of Cappell [Cap74] and Unil of Brookman–Khan [BK]. The advantages of these highly-connected theories are a decomposition of the structure set $\mathcal{S}_{\mathrm{CAT}}$ and a Mayer–Vietoris type sequence in L-theory.

By way of the L-theory of the NIL-category, we construct an isomorphism

$$LS^h_{*-1}(\Phi) \longrightarrow \widehat{H}^*(\mathbb{Z}_2; I) \oplus \mathrm{UNil}^h_{*+1}(\mathbb{Z}[\Phi]),$$

which relates the two theories. Thus we use algebraic surgery theory to prove this geometric observation of A. Ranicki [Ran81, §7.6].

Contents

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1. Introduction: $\mathcal{S}_{\mathrm{CAT}} \longrightarrow LS \longrightarrow \mathrm{UNil}$

Part 1. No surgeries required

2. Wall's embedding obstruction in LS

Reference: [Wal99, S11]
Choice of homotopy inverse of the homotopy equivalence.
Geometric cobordism groups $LS_{*-1}(\Phi)$.

3. Ranicki's isomorphism $LS \longrightarrow L(triad)$

Reference: [Ran81, §7.6]
Algebraic cobordism groups $LS_{*-1}(\mathbb{Z}[\Phi])$.
Transversality and quadratic construction for $LS_{*-1}(\Phi) \to LS_{*-1}(\mathbb{Z}[\Phi])$.
Thickening for $LS_{*-1}(\mathbb{Z}[\Phi]) \to L_{*+1}(triad)$.

4. An isomorphism $L(triad) \longrightarrow LNil$

Perform: One must study the appropriate covering space using chain complexes rather than homology modules and using homotopy-nilpotence rather than nilpotence. Also, some sort of Poincaré duality must be obtained. For the non-separating case of a $M \times S^1$, this was done in [Ran92, HR96].

5. Ranicki's isomorphism $LNil \longrightarrow L(\mathrm{NIL})$

Remark: The involution on NIL is due to J. Brookman [BK].

Reference/Perform: [Ran98, §35A]
Ranicki possibly shows that $LEnd$ is isomorphic to $L(\mathrm{END})$ for various endomorphism categories. This is unclear to me.

Part 2. Highly-connected theories

6. An isomorphism $L(\mathrm{NIL}) \longrightarrow \mathrm{Unil}$

Reference: [Ran89, §4]
In the general context of an additive category with involution, Ranicki proves the existence of highly-connected representatives and cobordisms.

Perform: Define an "additive instant obstruction" similar to [Ran89, §5].

7. Brookman's splitting obstruction in Unil

Reference: [BK]
Perform: Relate Wall's embedding obstruction to Brookman's splitting obstruction.

8. An isomorphism $\mathrm{Unil} \longrightarrow \mathrm{UNil}$

Remark: An explicit correlation of representatives and equivalence relations.

9. Cappell's splitting obstruction in UNil

References: [Cap74, Cap76]
Remark: Relate Brookman's obstruction to Cappell's obstruction.

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